ON SPACES ASSOCIATED WITH PRIMITIVES OF DISTRIBUTIONS IN ONE-SIDED HARDY SPACES

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Abstract. In this paper, we introduce the $\mathcal{H}^{p,+}_{q,\alpha}(w)$ spaces, where $0 < p \leq 1$, $1 < q < \infty$, $\alpha > 0$, and for weights w belonging to the class A^+_s defined by E. Sawyer. To define these spaces, we consider a one-sided version of the maximal function $N^+_{q,\alpha}(F,x)$ defined by A. Calderón. In the case that $w \equiv 1$, these spaces have been studied by A. Gatto, J. G. Jiménez and C. Segovia. We introduce a notion of p-atom in $\mathcal{H}^{p,+}_{q,\alpha}(w)$, and we prove that we can express the elements of $\mathcal{H}^{p,+}_{q,\alpha}(w)$ in term of series of multiples of p-atoms. On the other side, we prove that the Weyl fractional integral P_α can be extended to a bounded operator from the one-sided Hardy space $H^p_+(w)$ into $\mathcal{H}^{p,+}_{q,\alpha}(w)$. Moreover, we prove that this extension, if α is a natural number, is an isomorphism.

1. Notations, definitions and prerequisites

Let $f(x)$ be a Lebesgue measurable function defined on $\mathbb{R}$. The one-sided Hardy-Littlewood maximal functions $M^+f(x)$ and $M^-f(x)$ are defined as

$$M^+f(x) = \sup_{h>0} \int_x^{x+h} |f(t)|\,dt \text{ and } M^-f(x) = \sup_{h>0} \int_{x-h}^{x} |f(t)|\,dt.$$

As usual, a weight $w(x)$ is a measurable and non-negative function. If $E \subset \mathbb{R}$ is a Lebesgue measurable set, we denote its w-measure by $w(E) = \int_E w(t)dt$. A function $f(x)$ belongs to $L^s(w)$, $0 < s \leq \infty$, if $\|f\|_{L^s(w)} = \left(\int_{-\infty}^{\infty} f(x)^s w(x)dx\right)^{1/s}$ is finite.

A weight $w(x)$ belongs to the class A^+_s, $1 \leq s < \infty$, defined by E. Sawyer in [7], if there exists a constant c such that

$$\sup_{h>0} \left(\frac{1}{h}\int_{x-h}^{x} w(t)dt\right)\left(\frac{1}{h}\int_x^{x+h} w(t)^{-\frac{1}{s-1}}dt\right)^{s-1} \leq c,$$

for all real number x. We observe that $w(x)$ belongs to the class A^+_1 if $M^-w(x) \leq cw(x)$ for all real number x.

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Given $w(x) \in A_s^+$, $1 \leq s < \infty$, we can define two numbers $x_{-\infty}$ and $x_{+\infty}$, $-\infty \leq x_{-\infty} \leq x_{+\infty} \leq \infty$, such that

$$\begin{aligned} (i)\ w(x) &\equiv 0 \text{ in } (-\infty, x_{-\infty}), \\ (ii)\ w(x) &\equiv \infty \text{ in } (x_{+\infty}, \infty), \text{ and} \\ (iii) \quad 0 &< w(x) < \infty \text{ for almost every } x \in (x_{-\infty}, x_{+\infty}). \end{aligned}$$

In order to avoid the non-interesting case $x_{-\infty} = x_{+\infty}$, we assume that there exists a measurable set E satisfying $0 < w(E) < \infty$.

Let us fix $w \in A_s^+$ and let $x_{-\infty}$ be as before. Let $L^q_{loc}(x_{-\infty}, \infty)$, $1 < q < \infty$, be the space of the real-valued functions $f(x)$ on $\mathbb{R}$ that belong locally to L^q for compact subsets of $(x_{-\infty}, \infty)$). We endow $L^q_{loc}(x_{-\infty}, \infty)$ which the topology generated for the seminorms

$$|f|_{q,I} = \left(|I|^{-1} \int_I |f(y)|^q \, dy \right)^{1/q},$$

where $I = [a, b]$ is an interval in $(x_{+\infty}, \infty)$ and $|I| = b - a$.

For $f(x)$ in $L^q_{loc}(x_{-\infty}, \infty)$, we define a maximal function $n^+_{q,\alpha}(f; x)$ as

$$n^+_{q,\alpha}(f; x) = \sup_{\rho > 0} \rho^{-\alpha} \, |f|_{q,[x,x+\rho]} \, ,$$

where α is a positive real number.

Let N a non negative integer and $\mathcal{P}_N$ the subspace of $L^q_{loc}(x_{-\infty}, \infty)$ formed by all the polynomials of degree at most N. This subspace is of finite dimension and therefore a closed subspace of $L^q_{loc}(x_{-\infty}, \infty)$. We denote by E^q_N the quotient space of $L^q_{loc}(x_{-\infty}, \infty)$ by $\mathcal{P}_N$. If $F \in E^q_N$, we define the seminorm

$$\|F\|_{q,I} = \inf \left\{ |f|_{q,I} : f \in F \right\}.$$

The family of all these seminorms induces on E^q_N the quotient topology.

Given a real number $\alpha > 0$, we can write it $\alpha = N + \beta$, where N is a non negative integer and $0 < \beta \leq 1$. Now we fix $\alpha > 0$ and its decomposition $\alpha = N + \beta$ in the previous conditions.

For F in E^q_N, we define a maximal function $N^+_{q,\alpha}(F; x)$ as

$$N^+_{q,\alpha}(F; x) = \inf \left\{ n^+_{q,a}(f; x) : f \in F \right\}.$$

We say that an element F in E^q_N belongs to $\mathcal{H}^{p,+}_{q,\alpha}(w)$, $0 < p \leq 1$, if the maximal function $N^+_{q,\alpha}(F; x) \in L^p(w)$. The "norm" of F in $\mathcal{H}^{p,+}_{q,\alpha}(w)$ is defined as $\|F\|_{\mathcal{H}^{p,+}_{q,\alpha}(w)} = \left\| N^+_{q,\alpha}(F; x) \right\|_{L^p(w)}$.

Definition 1.1. *We shall say that a class $A \in E^q_N$ is a p-atom in $\mathcal{H}^{p,+}_{q,\alpha}(w)$ if there exist a representative $a(y)$ of A and an interval I (non necessarily bounded) such that*

i) $I \subset (x_{-\infty}, \infty)$, $w(I) < \infty$
ii) $\text{supp}(a) \subset I$
iii) $N^+_{q,\alpha}(A, x) \leq w(I)^{-1/p}$ for all $x \in (x_{-\infty}, \infty)$.

We shall say that I is an interval associated to the p-atom A.

Given a bounded function $f(y)$ with support in an interval $I = (x_{-\infty}, b]$ where b is finite and if $\alpha > 0$ we consider the Weyl fraccional integral

$$P_\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (y - x)^{\alpha - 1} f(y) dy \text{ for } x \in (x_{-\infty}, \infty),$$

where $\Gamma(\alpha)$ denotes the Gamma function. It is easy to see that $P_\alpha f(x) \in L^\infty_{loc}(x_{-\infty}, \infty)$ whenever $f \in L^\infty(x_{-\infty}, b]$. So, if $\alpha = N + \beta$ where $0 < \beta \leq 1$ and N is an integer, we denote $\overline{P_\alpha f}$ the class in E^q_N of the function $P_\alpha f(x)$.

As usual, $C^\infty_0(\mathbb{R})$ denotes the set of all functions with compact support having derivatives of all orders. We shall denote by $\mathcal{D}(x_{-\infty}, \infty)$ the space of all functions in $C^\infty_0(\mathbb{R})$ with support contained in $(x_{-\infty}, \infty)$ equipped with the usual topology and by $\mathcal{D}'(x_{-\infty}, \infty)$ the space of distributions on $(x_{-\infty}, \infty)$.

Given a positive integer γ and $x \in \mathbb{R}$, we shall say a function ψ in $C^\infty_0(\mathbb{R})$, belongs to the class $\Phi_\gamma(x)$ if there exists a bounded interval $I_\psi = [x, b]$ containing the support of ψ such that $D^\gamma \psi$ satisfies

$$|I_\psi| \, \|D^\gamma \psi\|_\infty \leq 1.$$

For $f \in \mathcal{D}'(x_{-\infty}, \infty)$ we define $f^*_{+,\gamma}(x)$ as

$$f^*_{+,\gamma}(x) = \sup \{ |\langle F, \psi \rangle| : \psi \in \Phi_\gamma(x) \},$$

for all $x > x_{-\infty}$. Let $w \in A^+_s$ and $0 < p \leq 1$. If γ is a natural number satisfying $(\gamma + 1) p \geq s > 1$ or $(\gamma + 1) p > 1$ if $s = 1$, then a distribution f in $\mathcal{D}'(x_{-\infty}, \infty)$ belongs to $H^p_{+,\gamma}(w)$ if the "p-norm" $\|f\|_{H^p_{+,\gamma}(w)} = \left(\int_{x_{-\infty}}^\infty f^*_{+,\gamma}(x)^p w(x) dx \right)^{1/p}$ es finite. These spaces have been defined by L. de Rosa and C. Segovia in [6]

A function $a(x)$ defined on $\mathbb{R}$ is called a p-atom in $H^p_{\gamma,+}(w)$ if there exists an interval I containing the support of $a(x)$, such that

(i) I is contained in $(x_{-\infty}, \infty)$, $w(I) < \infty$ and $\|a\|_\infty \leq w(I)^{-1/p}$

(ii) If the length of I is less than the distance $d(x_{-\infty}, I)$ from $x_{-\infty}$ to I, then

$$\int_I a(y) y^k dy = 0,$$

holds for every integer k, $0 \leq k < \gamma$.

The following theorem is of fundamental importance for the proof of Theorem 2.3 below.

Theorem 1.2. *Let $w \in A^+_s$, $\gamma \geq 1$ an integer and $0 < p \leq 1$ such that $(\gamma + 1) p \geq s > 1$ or $(\gamma + 1) p > 1$ if $s = 1$. Then, if $f \in H^p_{+,\gamma}(w)$ there exists a sequence $\{a_i\}$ of p-atoms in $H^p_{+,\gamma}(w)$ and a sequence $\{\lambda_i\}$ of real numbers such that $f = \sum \lambda_i a_i$ in $\mathcal{D}'(x_{-\infty}, \infty)$, and*

$$c_1 \|F\|^p_{H^p_{+,\gamma}(w)} \leq \sum |\lambda_i|^p \leq c_2 \|F\|^p_{H^p_{+,\gamma}(w)},$$

hold. Furthermore, the intervals associated to the p-atoms a_i can be assumed to be bounded.

For a proof see [6].

Let $F \in E^q_N$ and $f \in F$. Since f belongs to $L^q_{loc}(x_{-\infty}, \infty)$, $D^{N+1} f$ is defined in the sense of distributions. On the other hand, since any two representatives of F differ in a polynomial of degree at most N in $(x_{-\infty}, \infty)$, we get that $D^{N+1}f$ is independent of the representative $f \in F$ chosen. Therefore, for $F \in E_N^q$, we define $D^{N+1}F$ as the distribution $D^{N+1}f$, where f is any representative of F.

2. Statement of the main results

With the notation and definitions given in the section 1 we can state the main results of this paper.

Theorem 2.1 (Descomposition into atoms). *Let $w \in A_s^+$ and $0 < p \leq 1$, such that $(\alpha + 1/q)\, p \geq s > 1$ or $(\alpha + 1/q)\, p > 1$ if $s = 1$. Then, if $F \in \mathcal{H}_{q,\alpha}^{p,+}(w)$ there exists a sequence $\{\lambda_i\}$ of the number real and a sequence $\{A_i\}$ of p-atoms in $\mathcal{H}_{q,\alpha}^{p,+}(w)$ such that $F = \sum \lambda_i A_i$ en $E_N^q(x_{-\infty}, \infty)$. Moreover the series $\sum \lambda_i A_i$ converges in $\mathcal{H}_{q,\alpha}^{p,+}(w)$ and there exist two constants c_1 and c_2 such that*

$$c_1 \|F\|^p_{\mathcal{H}_{q,\alpha}^{p,+}(w)} \leq \sum |\lambda_i|^p \leq c_2 \|F\|^p_{\mathcal{H}_{q,\alpha}^{p,+}(w)} . \tag{1}$$

The next corollary proves that is enough to consider bounded intervals in the Definition 1.1.

Corrolary 2.2. *Under the hypotheses of Theorem 2.1 we can take the p-atoms $\{A_i\}$ in the decomposition having bounded associated intervals I_i.*

The following theorem shows that, in particular, if α is a natural number the spaces $H_{\gamma,+}^p(w)$ and $\mathcal{H}_{q,\alpha}^{p,+}(w)$ can be identified.

Theorem 2.3. *Let $w \in A_s^+, 0 < p \leq 1$, and $(\alpha + 1/q)\, p \geq s > 1$ or $(\alpha + 1/q)\, p > 1$ if $s = 1$. Let γ an integer such that $\gamma \geq \alpha$. Then, P_α*

can be extended to a bounded linear operator from $H_{\gamma,+}^p(w)$ into $\mathcal{H}_{q,\alpha}^{p,+}(w)$. Moreover, if we suppose that α is a natural number this extension is an isomorphism.

3. Some previous lemmas

The next lemma contains the basic results for A_s^+ weights and one-sided maximal functions that we shall need in this paper.

Lemma 3.1.

1. Let $1 \leq s_1 < s_2 < \infty$. if the weight belongs to the class $A_{s_1}^+$, then it also belongs to $A_{s_2}^+$.
2. Let $1 < s < \infty$. The one-sided Hardy-Littlewood maximal M^+ is bounded on $L^s(w)$ if and only if w belongs to A_s^+.
3. Let $w \in A_s^+, 1 \leq s < \infty$. Let $a < b$ be the end points of the bounded interval I. Then, the interval I^- with end points $a - |I|$ and a, satisfies
$$w(I^-) \leq c_w w(I),$$
where the constant c_w does not depend on I.
4. Given $w \in A_s^+$, $1 \leq s < \infty$ for every $a \in \mathbb{R}$, the $w-$measure of the interval (a, ∞) is equal to infinite.

(5) Let $1 < s < \infty$. Then, if $w \in A_s^+$ there exists $\varepsilon > 0$ such that $w \in A_{s-\varepsilon}^+$.

Proofs of parts (2) and (5) may be found in [7] and [5]. Proofs of parts (1) and (4) are very simple and shall be omitted. Part (3) is an immediate consequence of (2).

Lemma 3.2. *There exists an infinitely differentiable function ϕ with support in $[-1,0]$, such that*

$$P(x) = \int \lambda P(y)\phi(\lambda[x-y])dy,$$

for all polynomials $P(x)$ of degree less than or equal to N and for every $\lambda > 0$.

The proof is the same as Lemma 2.6 in [2].

Lemma 3.3. *Let f_1 and f_2 be two representatives of an element F in E_N^q, and $P(y) = f_1(y) - f_2(y)$. There exists a constant c_k such that*

$$\left|\left(\frac{d}{dy}\right)^k P(y)\right| \leq c_k \left(n_{q,a}^+(f_1,x_1) + n_{q,a}^+(f_2,x_2)\right)\left(|x_1-y| + |x_2-y|\right)^{\alpha-k}$$

holds for every x_1, x_2 and y in $(x_{-\infty}, \infty)$.

Proof. we assume that $x_2 \geq x_1$. First, we suppose that $y \geq x_2$, in this case, by Lemma 3.2 and proceeding as in the proof of Lemma 1 in [1], we get

$$\text{(2)} \qquad \left|\left(\frac{d}{dy}\right)^k P(y)\right| \leq c\left(n_{q,a}^+(f_1,x_1) + n_{q,a}^+(f_2,x_2)\right)(y - x_1 + y - x_2)^{\alpha-k}.$$

Now, if $y \leq x_2$, taking into account that $D^k(f_1(y) - f_2(y))$ is a polynomial of degree at most $N-k$, and by its Taylor's expansion, we have

$$\text{(3)} \qquad D^k(f_1(y) - f_2(y)) = \sum_{j=0}^{N-k} D_y^{k+j}(f_1(y) - f_2(y))\Big|_{y=x_2} \frac{(y-x_2)^j}{j!}.$$

Using (2) with $y = x_2$, we obtain

$$\begin{aligned}&\left|D_y^{k+j}(f_1(y) - f_2(y))\Big|_{y=x_2}\right| \\ &\leq c\left(n_{q,a}^+(f_1,x_1) + n_{q,a}^+(f_2,x_2)\right)|x_1 - x_2|^{\alpha-k-j}.\end{aligned}$$

Then,

$$\text{(4)} \qquad \begin{aligned}&\left|D^k(f_1(y) - f_2(y))\right| \\ &\leq C\left(n_{q,a}^+(f_1,x_1) + n_{q,a}^+(f_2,x_2)\right)(x_2 - x_1)^{\alpha-k} \sum_{j=0}^{N-k}\left(\frac{|y-x_2|}{x_2 - x_1}\right)^j\end{aligned}$$

Since $|x_1 - x_2| \leq |y - x_1| + |y - x_2|$, we obtain that the right side of (4) is bounded by

$$C_N\left(n_{q,a}^+(f_1,x_1) + n_{q,a}^+(f_2,x_2)\right)(x_2 - x_1)^{\alpha-k}\left(\frac{|y-x_2| + |y-x_1|}{x_2 - x_1}\right)^{N-k}.$$

From this fact and taking into account that $\alpha = N + \beta$, we obtain that

$$\left|D^k(f_1(y) - f_2(y))\right| \leq c\left(n^+_{q,a}(f_1, x_1) + n^+_{q,a}(f_2, x_2)\right)(|y - x_1| + |y - x_2|)^{\alpha - k},$$

holds for every y. ■

Lemma 3.4. *Let F belongs to E^q_N with $N^+_{q,\alpha}(F, x_0) < \infty$. Then:*

(i) there exists a unique f in F such that $n^+_{q,a}(f, x_0) < \infty$ and, therefore, $N^+_{q,\alpha}(F; x_0) = n^+_{q,a}(f, x_0)$.

(ii) For any interval $I = [a, b] \subset (x_{-\infty}, \infty)$ with $a \geq x_0$, there exists a constant c depending on x_0 and I such that if f is the unique representative of F given in (i), then

$$\|F\|_{q,I} \leq |f|_{q,I} \leq c\, n^+_{q,a}(f, x_0) = c\, N^+_{q,\alpha}(F, x_0)$$

The constant c can be chosen independently of x_0 provided that x_0 varies in a compact set.

Proof. The proof is similar to that of Lemma 3 in [3]. □

Corrolary 3.5. *If $\{F_i\}$ is a sequence of elements in E^q_N converging to F in $\mathcal{H}^{p,+}_{q,\alpha}(w)$, $0 < p \leq 1$, then $\{F_i\}$ converges to F in E^q_N.*

Proof. Let an interval $I = [a, b] \subset (x_{-\infty}, \infty)$. Since $a > x_{-\infty}$, $d(x_{-\infty}, a) = r > 0$. Let n be the first positive integer such that $\frac{|I|}{n} < \frac{r}{2}$, and we consider the interval $I^-_n = \left[a - \frac{|I|}{n}, a\right]$. Now, for (ii) of Lemma 3.4

$$\|F - F_i\|_{q,I} \leq C_I N^+_{q,\alpha}(F - F_i; x) \text{ for every } x \text{ in } I^-_n. \tag{5}$$

On the other hand, since $I^-_n \subset (x_{-\infty}, \infty)$, $w(I^-_n) > 0$. Then

$$\|F - F_i\|^p_{q,I} \leq C_I w(I^-_n)^{-1} \int_{I^-_n} N^+_{q,\alpha}(F - F_i, x)^p w(x)dx \leq c_{I,w}\ \|F - F_i\|^p_{\mathcal{H}^{p,+}_{q,\alpha}(w)},$$

which proves the corollary. ■

Lemma 3.6. *If $\{F_i\}$ is a sequence of elements in E^q_N such that the series $\sum_i N^+_{q,\alpha}(F_i, x)$ is finite a.e. x in $(x_{-\infty}, \infty)$. Then*

(i) The series $\sum_i F_i$ converges in E^q_N to an element F and

$$N^+_{q,\alpha}(F; x) \leq \sum_i N^+_{q,\alpha}(F_i, x) \text{ for all } x \in (x_{-\infty}, \infty). \tag{6}$$

(ii) Let x_0 be a point where $\sum_i N^+_{q,\alpha}(F_i; x_0)$ is finite. If f_i is the unique representative of F_i satisfying $n^+_{q,a}(f_i; x_0) = N^+_{q,\alpha}(F_i; x_0)$, then $\sum_i f_i$ converges in $L^q_{loc}(x_{-\infty}, \infty)$ to a function f that is the unique representative of F satisfying $n^+_{q,a}(f; x_0) = N^+_{q,\alpha}(F; x_0)$.

Proof. First we prove (ii). Let x_0 be a point where $\sum_i N^+_{q,\alpha}(F_i; x_0)$ is finite. Then, by Lemma 3.4, for each i there exists a representative f_i of F_i satisfying $n^+_{q,a}(f_i; x_0) = N^+_{q,\alpha}(F_i; x_0) < \infty$. Let $x \in (x_{-\infty}, \infty)$ another point such that $\sum_i N^+_{q,\alpha}(F_i; x)$ is finite, then for each i there exists a polynomial $P_i(x, y)$ of degree at most N such that $n^+_{q,a}(f_i(y) - P_i(x, y); x) = N^+_{q,\alpha}(F_i; x)$. Using Lemma 3.3, we get

$$|P_i(x, y)| \leq \left(N^+_{q,\alpha}(F_i; x_0) + N^+_{q,\alpha}(F_i; x)\right)(|y - x_0| + |y - x|)^{\alpha}. \tag{7}$$

Let us fix an interval $I = [a, b] \subset (x_{-\infty}, \infty)$ and we consider $x_1 \le a$ such that $\sum_i N^+_{q,\alpha}(F_i; x_1) < \infty$. Then, by (ii) of Lemma 3.4 and (7), we obtain

$$|f_i|_{q,I} \le |f_i - P_i(x_1, .)|_{q,I} + |P_i(x_1, .)|_{q,I}$$
$$\le C_{I,x_1,x_0}\left(N^+_{q,\alpha}(F_i; x_0) + N^+_{q,\alpha}(F_i; x_1)\right).$$

Thus

$$\left|\sum_{i=1}^{k} f_i - \sum_{i=1}^{m} f_i\right|_{q,I} \le \sum_{i=m+1}^{k} |f_i|_{q,I} \le C_{I,x_1,x_0} \sum_{i=m+1}^{k} \left(N^+_{q,\alpha}(F_i; x_0) + N^+_{q,\alpha}(F_i; x_1)\right),$$

which proves that there exists f en $L^q_{loc}(x_{-\infty}, \infty)$ such that $\sum_{i=1}^{\infty} f_i = f$ in this space. Let us denote by F the class of f in E^q_N. Since

$$n^+_{q,\alpha}(f; x_0) \le \sum_i n^+_{q,\alpha}(f_i; x_0) = \sum_i N^+_{q,\alpha}(F_i; x_0) < \infty, \tag{8}$$

we have that f is the unique representative of F satisfying $N^+_{q,\alpha}(f; x_0) = N^+_{q,\alpha}(F; x_0)$.

Now, we will prove (i). As consequence of inequality

$$\left\|\sum_{i=1}^{K} F_i - F\right\|_{q,I} \le \left|\sum_{i=1}^{K} f_i - f\right|_{q,I},$$

we have $F = \sum_{i=1}^{\infty} F_i$ in E^q_N. Moreover, from (8) we obtain

$$N^+_{q,\alpha}(F; x_0) \le \sum_i N^+_{q,\alpha}(F_i; x_0).$$

This conclude the proof. ■

Corrolary 3.7. *The space $\mathcal{H}^{p,+}_{q,\alpha}(w)$, $0 < p \le 1$, is complete.*

Taking into account Lemma 3.6, the proof of this result is similar to that of Corollary 2 in [3].

Lemma 3.8. *The maximal function $N^+_{q,a}(F, x)$ associated with a class F in E^q_N is lower semicontinuous.*

See Lemma 6 in [1].

Lemma 3.9. *Let f a representative of F in E^q_N. We suppose that $N^+_{q,\alpha}(F; x)$ is finite and we denote by $P(x, y)$ the unique polynomial of degree almost N such that $n^+_{q,\alpha}(f(y) - P(x, y); x) = N^+_{q,\alpha}(F; x)$. Then $f(x) = P(x, x)$ for almost every x such that $N^+_{q,\alpha}(F; x)$ is finite.*

See Lemma 2 in [1].

Lemma 3.10. *Let F in E^q_N. We suppose that $N^+_{q,\alpha}(F, x) \le t$ for every x belonging to a set $E \subset (x_{-\infty}, \infty)$. Let f a representative of F and $P(x, y)$ the unique polynomial in $\mathcal{P}_N$ such that $N^+_{q,\alpha}(F; x) = n^+_{q,\alpha}(f(y) - P(y, x); x)$. For each x in E we define $A_k(x) = D^k_y P(x, y)\big|_{y=x}$. Then,*

$$\left|A_k(x) - \sum \frac{1}{i!}(x - \overline{x})^i A_{k+i}(\overline{x})\right| \le c\, t\ |x - \overline{x}|^{\alpha-k}, \tag{9}$$

for all x and $\overline{x}$ in E. Furthermore, A_N satisfies a Lipschitz-β condition on E with constant ct (we recall that $\alpha = N + \beta$, where $0 < \beta \leq 1$) and if $0 \leq k < N$, $A_k(x)$ satisfies a uniform Lipschitz-1 condition on every bounded subset of E.

By Lemma 3.2 and proceeding as in the proof of Lemma 5 in [1] we obtain the proof of this lemma.

The next results will be used in the proof of Theorem 2.3.

Lemma 3.11. *Let $f \in L^\infty$, with supp$(f) \subset I = (x_{-\infty}, b]$ where b is finite.*

Then

$$N^+_{q,\alpha}(\overline{P_\alpha f}; x) \leq C \|f\|_\infty \text{ for every } x \in (x_{-\infty}, \infty), \tag{10}$$

where C does not depend on f.

Proof. For $x \in (x_{-\infty}, \infty)$ and $z > 0$, we define

$$R(x,z) = \tag{11}$$

$$\frac{1}{\Gamma(\alpha)}\left[\int_{x+z}^{\infty} (y-x-z)^{\alpha-1} f(y)dy - \int_x^\infty \left(\sum_{k=0}^N c_{k,\alpha} (y-x)^{\alpha-1-k} z^k\right) f(y)dy\right],$$

where $\left(\sum_{k=0}^N c_{k,\alpha} (y-x)^{\alpha-1-k} z^k\right)$ is the Taylor's expansion of order N of the function $(y-x-z)^{\alpha-1}$. Let $\rho > 0$. We will estimate $\rho^{-\alpha} |R(x,.)|_{q,[0,\rho]}$. We recall that $\alpha = N + \beta$, and we consider first the case $0 < \beta < 1$. Since

$$R(x,z) = \frac{1}{\Gamma(\alpha)} \int_{x+z}^{x+2z} (y-x-z)^{\alpha-1} f(y)dy$$

$$+\frac{1}{\Gamma(\alpha)} \int_{x+2z}^{\infty} \left[(y-x-z)^{\alpha-1} - \sum_{k=0}^N c_{k,\alpha} (y-x)^{N-k} z^k\right] f(y)dy$$

$$-\frac{1}{\Gamma(\alpha)} \sum_{k=0}^N c_{k,\alpha} \int_x^{x+2z} (y-x)^{\alpha-1-k} f(y)dy\, z^k = A_1 + A_2 - A_3. \tag{12}$$

For A_2, by the mean value Theorem and since $\beta < 1$, we obtain

$$|A_2| \leq C \|f\|_\infty \int_{x+2z}^\infty (y-x)^{\beta-2}\, dy\; z^{N+1} \leq C \|f\|_\infty\; z^\alpha,$$

For A_3, we have

$$|A_3| \leq C \|f\|_\infty \sum_{k=0}^N \int_x^{x+2z} (y-x)^{\alpha-1-k}\, dy\; z^k \leq C \|f\|_\infty \sum_{k=0}^N z^{\alpha-k} z^k \leq C \|f\|_\infty z^\alpha. \tag{13}$$

In the same way we obtain

$$|A_1| \leq C \|f\|_\infty z^\alpha.$$

Then, for $\beta < 1$, we have that

$$|R(x,z)| \leq C \|f\|_\infty z^\alpha, \tag{14}$$

however (14) also holds for $\beta=1$. In fact, since $(y-x-z)^N$ is a polynomial of degree N, it coincides with its Taylor's expansion of order N, so we have that

$$R(x,z)=$$

$$\frac{1}{\Gamma(N+1)}\left[\int_{x+z}^{\infty}(y-x-z)^N f(y)dy-\int_x^{\infty}\left(\sum_{k=0}^{N}c_{k,N}(y-x)^{N-k}f(y)\,z^k\right)dy\right]$$

$$=-\frac{1}{\Gamma(N+1)}\sum_{k=0}^{N}c_{k,N}\int_x^{x+z}(y-x)^{N-k}f(y)dy\,z^k.$$

Then , in the same way that we obtained (13), we can prove that (14) also holds in this case.

As consequence of (14) we obtain

$$\rho^{-\alpha}\left|R(x,.)\right|_{q,[0,\rho]}=\rho^{-\alpha}\left(\frac{1}{\rho}\int_0^{\rho}|R(x,z)|^q\,dz\right)^{1/q}$$

$$\leq C\left\|f\right\|_{\infty}\rho^{-\alpha}\left(\frac{1}{\rho}\int_0^{\rho}z^{\alpha q}dz\right)^{1/q}\leq C\left\|f\right\|_{\infty},$$

which implies (10). ■

Lemma 3.12. *Let $a(y)$ a p-atom in $H^p_{+,\gamma}(w)$ with vanishing moments up to the order N. Furthermore, we suppose that supp$(a)\subset I=[x_0,b]$ and $\|a\|_\infty\leq w(I)^{-1/p}$. Then, for every natural number k and if $x\in(x_{-\infty},\infty)$ $\cap\{|x-x_0|>2|I|\}$ holds that*

$$\left|D^kP_\alpha a(x)\right|\leq C_k\,w(I)^{-1/p}\frac{|I|^{N+2}}{|x|^{2+k-\beta}}$$

Proof. Without loss of generality, we can suppose that $I=[0,b]$. As supp$(a)\subset I$, the result is trivial if $x>2b$, then we consider $x<-2b$. In this case $P_\alpha a(x)=\frac{1}{\Gamma(\alpha)}\int_0^b(x-y)^{\alpha-1}a(y)dy$ and

$$D^kP_\alpha a(x)=c_{k,\alpha}\int_0^b(y-x)^{\alpha-1-k}a(y)dy.$$

Then, taking into account that $a(y)$ has vanishing moments up to order N, $x<-2b$, and recalling that $\alpha=N+\beta$, we have

$$\begin{aligned}\left|D^kP_\alpha a(x)\right| &= \left|c_{k,\alpha}\int_0^b\left((y-x)^{\alpha-1-k}-\sum_{i=0}^{N}c_{\alpha,i}(-x)^{\alpha-1-k-i}y^i\right)a(y)dy\right|\\ &\leq C\,|x|^{\alpha-k-N-2}\int_0^b|a(y)|\,|y|^{N+1}dy\leq Cw(I)^{-1/p}\frac{b^{N+2}}{|x|^{2+k-\beta}}.\end{aligned}$$

■

4. Proof of the results

The following lemma states some properties of a one-sided partition of unity that can be found in [6].

Lemma 4.1 (a one-sided partition of unity)**.** *Let $a<b$ and we consider the interval $I=(a,b)$. Then there exists a sequence $\{\eta_j\}_{j=1}^{\infty}$ of C_0^∞ functions satisfying the following conditions*

1) $0 \le \eta_j(x) \le 1$ and $\sum_j \eta_j(x)\chi_{(a,b)}(x) = \chi_{(a,b)}(x)$.
2) For each positive integer j, if $I_j = [a + 2^{-j}(b-a), a + 2^{-j+2}(b-a)]$ $\operatorname{supp}(\eta_j) \subset I_j$. If we denote $r_j = \frac{(b-a)}{2^j}$, then for every $x \in I_j$ $r_j \le x - a \le cr_j$, where c does not depend on j.
3) If $\hat{I}_j = \left(a + 2^{-j-1}(b-a), \min\{a + 2^{-j+2}(b-a), b\}\right)$, $\cup_j \hat{I}_j = I$. Furthermore, the number of interval $\hat{I}_j$ that intersect to other interval $\hat{I}_k$ does not exceed two.
4) If k is an integer, $k \ge 0$, we have
$$\left| D^k \eta_j(x) \right| \le C_k r_j^{-k},$$
where C_k does not depend on j.

Let $F \in \mathcal{H}^{p,+}_{q.\alpha}(w)$. Given $t > 0$, we consider
$$\Omega = \Omega_t = \{x \in (x_{-\infty}, \infty) : N^+_{q,\alpha}(F, x) > t\},$$
since $N^+_{q,\alpha}(F, x) \in L^p(w)$, $w(\Omega) < \infty$ and by Lemma 3.8 this is an open subset of $(x_{-\infty}, \infty)$. Then $\Omega = \bigcup_{i=1}^{\infty} I_i$, where intervals $I_i = (a_i, b_i)$ are the connected components of Ω. We observe that $b_i < \infty$, since $w(I_i) \le w(\Omega) < \infty$ (see part (3) of Lemma 3.1). If there is an interval I_i with $a_i = x_{-\infty}$, then we will assume that $i = 1$. If not, we shall assume that $I_1 = \emptyset$. Let f belonging We define
$$\theta_1(y) = \chi_{I_1}(y)(f(y) - P(b_1, y)), \tag{15}$$
where $P(b_1, y) \in \mathcal{P}_N$ and $N^+_{q,\alpha}(F, b_1) = n_{q,\alpha}(f(y) - P(b_1, y))$.

On the other hand, for each $i > 1$, Let $\{\eta_{i,j}\}_{i>1,j\ge1}$ be the partition of unity as Lemma 4.1 associated with each interval $I_i = (a_i, b_i)$ and we denote $I_{i,j}$ and $\hat{I}_{i,j}$ that intervals I_j and $\hat{I}_j$ of the same lemma. We define $x_{i,j} = b_i$ if $j = 1, 2$ and $x_{i,j} = a_i$ for $j > 2$. Let $\mathcal{C} = (x_{-\infty}, \infty) - \Omega$. We observe that each point $x_{i,j}$ satisfies $d(\hat{I}_{i,j}, \mathcal{C}) = d(\hat{I}_{i,j}, x_{i,j})$ where $\hat{I}_{i,j} = \left(a_i + 2^{-j-1}(b_i - a_i), \min\left\{a_i + 2^{-j+2}(b_i - a_i), b_i\right\}\right)$. Furthermore, as the points $x_{i,j}$ belong to $\mathcal{C}$, we have that $N^+_{q,\alpha}(F, x_{i,j}) \le t$. We denote $P(x_{i,j}, y)$ the polynomial satisfying $N^+_{q,\alpha}(F, x_{i,j}) = n^+_{q,\alpha}(f(y) - P(x_{i,j}, y))$. Now, for each $i > 1$ and $j \ge 1$, we define
$$\theta_{i,j}(y) = \eta_{i,j}(y)\chi_{I_i}(y)\left(f(y) - P(x_{i,j}, y)\right). \tag{16}$$
The functions $\theta_{i,j}$ and θ_1 belong to $L^q_{loc}(x_{-\infty}, \infty)$. Let us denote by $\Theta_{i,j}$ and Θ_1 the class of $\theta_{i,j}$ and θ_1 in E^q_N respectively.

For the following two lemmas we will use the previous notation.

Lemma 4.2. *Let $F \in \mathcal{H}^{p,+}_{q,\alpha}(w)$, and $f \in F$. If $g(y)$ is defined in $(x_{-\infty}, \infty)$ as*
$$g(y) = \begin{cases} \sum_{i=2}^{\infty}\sum_{j=1}^{\infty} \eta_{i,j}(y)\chi_{I_i}(y)P(x_{i,j}, y) + \chi_{I_1}(y)P(b_1, y)) & \text{if } y \in \Omega, \\ f(y) & \text{if } y \notin \Omega, \end{cases}$$
and G denote its class in E^q_N, then there exists a constant C such that
$$N^+_{q,\alpha}(G, x) \le Ct \text{ for all } x \in (x_{-\infty}, \infty).$$

Proof. It will be enough to prove that the function g agrees almost everywhere with a function having derivatives continuous up to order N and its derivative of order N satisfies a Lipschitz-β condition with constant ct on $(x_{-\infty},\infty)$. The function $g(y)$ is infinitely differentiable on Ω, and if $x\in\Omega$, we have that

$$D^k g(x) = \tag{17}$$

$$\sum_{i=2}^{\infty}\sum_{j=1}^{\infty}\sum_{h=0}^{k}\frac{k!}{h!k-h!}D^t\eta_{i,j}(x)\;D_y^{k-h}P(x_{i,j},y)\Big|_{y=x}+\chi_{I_1}(x)\;D_y^{k-h}P(b_1,y)\Big|_{y=x},$$

Let $\widetilde{x}\in\mathcal{C}$. By condition (3) of Lemma 4.1 we have, for x in Ω, that

$$D_y^kP(\widetilde{x},y)\Big|_{y=x} = \tag{18}$$

$$\sum_{i=2}^{\infty}\sum_{j=1}^{\infty}\sum_{h=0}^{k}\frac{k!}{h!(k-h)!}D^h\eta_{i,j}(x)\;D_y^{k-h}P(\widetilde{x},y)\Big|_{y=x}+\chi_{I_1}(x)\;D_y^{k}P(\widetilde{x},y)\Big|_{y=x}$$

Let $\overline{x}\in\mathcal{C}=(x_{-\infty},\infty)-\Omega$, $x\in\Omega$ and we denote $\widetilde{x}$ the point in $\mathcal{C}$ closest to x. From (17) and (18), we obtain for $x\in\Omega$ that

$$D^kg(x)-D_y^kP(\overline{x},y)\Big|_{y=x} = \tag{19}$$

$$\sum_{i=2}^{\infty}\sum_{j=1}^{\infty}\sum_{h=0}^{k}\frac{k!}{h!(k-h)!}D^h\eta_{i,j}(x)\left[D_y^{k-h}P(x_{i,j},y)\Big|_{y=x}-D_y^{k-h}P(\widetilde{x},y)\Big|_{y=x}\right]$$

$$+\chi_{I_1}(x)\left[D_y^kP(b_1,y)\Big|_{y=x}-D_y^kP(\widetilde{x},y)\Big|_{y=x}\right]+D_y^kP(\widetilde{x},y)\Big|_{y=x}-D_y^kP(\overline{x},y)\Big|_{y=x}.$$

We suppose that $x\in\widehat{I}_{i,j}=\left(a_i+2^{-j}(b_i-a_i),\min\left\{a_i+2^{-j+2}(b_i-a_i),b_i\right\}\right)$ for some $i>1$ and $j\geq 1$. We denote $r_{i,j}=\frac{b_i-a_i}{2^j}=\frac{|I_i|}{2^j}$. Since $x_{i,1}=x_{i,2}=b_i$ and $x_{i,j}=a_i$ for $j>2$, and taking into account (2) of Lemma 4.1, we have that

$$|\widetilde{x}-x|\leq|x_{i,j}-x|\leq cr_{i,j},\ \text{and}$$
$$|\widetilde{x}-x|\leq|x_{i,j}-x|\leq c\,|\overline{x}-x|\,.$$

By Lemma 3.3 and since $x_{i,j}$, $\widetilde{x}$, and $\overline{x}$ belong to $\mathcal{C}$, we obtain

$$\left|\;D_y^{k-h}P(x_{i,j},y)\Big|_{y=x}-D_y^{k-h}P(\widetilde{x},y)\Big|_{y=x}\right|\leq ct\;|\overline{x}-x|^{\alpha-k}\,r_{i,j}^h,\qquad\text{and}$$

$$\left|D_y^kP(\widetilde{x},y)\Big|_{y=x}-D_y^kP(\overline{x},y)\Big|_{y=x}\right|\leq c\,t\,(|\overline{x}-x|)^{\alpha-k}\,. \tag{20}$$

Applying These estimates in (19), and using condition (4) of Lemma 4.1, we have that

$$\left|D^kg(x)-D_y^kP(\overline{x},y)\Big|_{y=x}\right|\leq ct\,|\overline{x}-x|^{\alpha-k}\,. \tag{21}$$

If $x\in I_1=(x_{-\infty},b_1)$, we have that $\widetilde{x}=b_1$. Since in the right member of (19) all the terms are cancelled except the last one, by (20) we have that (21) also holds in this case.

Now, we take $k = N+1$. Assuming that $x \in I_i$ for $i > 1$, from (19), we obtain

$$D^{N+1}g(x) = \tag{22}$$

$$\sum_{i=2}^{\infty}\sum_{j=1}^{\infty}\sum_{h=1}^{N+1} \frac{N+1!}{h!(N+1-h)!} D^h\eta_{i,j}(x)\left[D_y^{N+1-h}(P(x_{i,j},y) - P(\widetilde{x},y))\Big|_{y=x}\right].$$

Since $x \in I_i$ where $i > 1$, then x belongs to $\widehat{I}_{i,j_0}$ for some $j_0 \geq 1$. We suppose that $j_0 > 5$. Then $D^h\eta_{i,j}(x) = 0$ for $j = 1,2,3$. Moreover, for $j > 3$, $\widetilde{x} = x_{i,j} = a_i$, then $D^{N+1}g(x)$ is vanished. On the other hand, if $j_0 \leq 5$, then $x \notin \widehat{I}_{i,j}$ for $j > 7$. Furthermore if $j \leq 7$, we have that $r_{i,j} \geq 2^{-8}|I_i|$. From the last estimate and by condition 4) of Lemma 4.1, we obtain

$$\left|D^h\eta_{i,j}(x)\right| \leq c\,|I_i|^{-h}.$$

Now, applying Lemma 3.3 and recalling that $\alpha = N+\beta$, we obtain

$$\left| D_y^{N+1-h}P(x_{i,j},y)\Big|_{y=x} - D_y^{N+1-h}P(\widetilde{x},y)\Big|_{y=x}\right| \leq ct\; r_{i,j}^{\beta-1+h} \leq ct\;\; |I_i|^{\beta-1+h}.$$

From (22) and taking into account these estimates, we obtain

$$\left|D^{N+1}g(x)\right| \leq c\,t\;\, |I_i|^{\beta-1} \text{ for every } x \in I_i. \tag{23}$$

If $x \in I_1$, $g(x) = P(b_1,x)$, and therefore in this case $D^{N+1}g(x) = 0$ and (23) also holds. Now, for each $k = 0,1,2,...N+1$, we define the function B_k in $(x_{-\infty},\infty)$ as

$$B_k(x) = \begin{cases} D^kg(x) & \text{si } x \in \Omega \\ A_k(x) & \text{si } x \in \mathcal{C} \end{cases},$$

where $A_k(x) = D_y^kP(x,y)\big|_{y=x}$ is the function of Lemma 3.10. Then, if $x \in \Omega$, and $\overline{x} \in \mathcal{C}$ the inequality (21) can be rewritten as

$$\left|B_k(x) - \sum_{h=0}^{N-k} B_{k+h}(\overline{x})\frac{(x-\overline{x})^h}{h!}\right| \leq c\;t\;\; |\overline{x}-x|^{\alpha-k}, \tag{24}$$

para $0 \leq k \leq N$. Now since $N_{q,\alpha}^{+}(F.x) \leq t$ in $\mathcal{C}$, Lemma 3.10 shows that this inequality holds also for $x \in \mathcal{C}$. This shows that $B_k(x)$ is continuous for $0 \leq k \leq N$, and for x in $\mathcal{C}$. Furthermore, for $1 \leq k \leq N$, $B_k(x)$ is continuous in $(x_{-\infty},\infty)$ since $B_k(x) = D^kg(x)$ in Ω. By (24) with $k = N$ we obtain that B_N satisfies

$$|B_N(x) - B_N(y)| \leq c\;t\;\; |x-y|^{\beta}, \tag{25}$$

for every x and $y \in (x_{-\infty},\infty)$ and one of them in $\mathcal{C}$. Now we will prove that (25) also holds without every x and y in $(x_{-\infty},\infty)$. We consider $x_1 < x_2$ in Ω, then $x_1 \in I_{i_1}$, and $x_2 \in I_{i_2} = (a_{i_2}, b_{i_2})$. If $i_1 \neq i_2$, we have

$$|x_1 - a_{i_2}|^{\beta} + |x_2 - a_{i_2}|^{\beta} \leq 2\,|x_2 - x_1|^{\beta}. \tag{26}$$

Taking into account that $a_{i_2} \in \mathcal{C}$, using (25) and (26), we obtain

$$\begin{aligned} |B_N(x_1) - B_N(x_2)| &\leq |B_N(x_1) - B_N(a_{i_2})| + |B_N(x_1) - B_N(a_{i_2})| \\ &\leq ct\left[|x_1 - a_{i_2}|^{\beta} + |x_2 - a_{i_2}|^{\beta}\right] \\ &\leq ct\,|x_1 - x_2|^{\beta}. \end{aligned} \tag{27}$$

On the other hand, if $i_2 = i_1$, i.e., x_1 and x_2 in I_{i_1}. Then, taking into account that $B_{N+1}(x) = D^{N+1}g(x)$ for $x \in \Omega$, (23) and the inequality $|x_1 - x_2| \leq |I_{i_1}|$, we have

$$|B_N(x_1) - B_N(x_2)| \leq |B_{N+1}(\zeta)|\, |x_1 - x_2| \leq ct\frac{|x_1 - x_2|}{|I_{i_1}|^{1-\beta}}$$

$$\leq ct\, |x_1 - x_2|^{\beta}. \tag{28}$$

As consequence (25), (27) and (28), we obtain that B_N satisfies a Lipschitz-β condition in $(x_{-\infty}, \infty)$ with constant ct. The inequality (24) shows that $D^k B_0(x) = B_k(x)$ in $\mathcal{C}$, identity which also holds in Ω. Furthermore, Lemma 3.9 permits us assert that $B_0(x) = A_0(x) = P(x, x) = g(x)$ almost everywhere in $\mathcal{C}$. Thus we conclude that $g(x)$ coincides almost everywhere in $(x_{-\infty}, \infty)$ with $B_0(x)$ which has continuous derivatives up to order N in $(x_{-\infty}, \infty)$, and its derivative of order N satisfies a Lipschitz-β condition with constant ct. ■

With the notation given in (16) we have the following result.

Lemma 4.3 (one-sided Calderón-Zygmund-type)**.** *Let $F \in \mathcal{H}^{p,+}_{q,\alpha}(w)$ and $w \in A^+_s$, where $(\alpha + 1/q)\, p \geq s > 1$ or $(\alpha + 1/q)\, p > 1$ if $s = 1$. Then, the following conditions are satisfied:*

(i) If $x \in \hat{I}_{i,j} = \left(a_i + 2^{-j-1}(b_i - a_i), \min\left\{a_i + 2^{-j+2}(b_i - a_i), b_i\right\}\right)$

$$\begin{aligned} N^+_{q,\alpha}(\Theta_{i,j}, x) &\leq CN^+_{q,\alpha}(F, x), \text{ and} \\ N^+_{q,\alpha}(\Theta_1, x) &\leq CN^+_{q,\alpha}(F, x)\chi_{I_1}(x) \text{ for all } x \in (x_{-\infty}, \infty) \end{aligned}$$

(ii) If $x > x_{-\infty}$ and $x \notin \hat{I}_{i,j}$

$$N^+_{q,\alpha}(\Theta_{i,j}, x) \leq ct\left[M^+\chi_{\hat{I}_{i,j}}(x)\right]^{\alpha+1/q}.$$

(iii) The series $\sum_{i,j} N^+_{q,\alpha}(\Theta_{i,j}; x) + N^+_{q,\alpha}(\Theta_1; x)$ is pointwise convergent for almost every x in $(x_{-\infty}, \infty)$. Moreover,

$$\int \left(\sum_{i>1,j} N^+_{q,\alpha}(\Theta_{i,j}; x) + N^+_{q,\alpha}(\Theta_1; x)\right)^p w(x)dx \leq c \int_{\Omega} N^+_{q,\alpha}(F, x)\ ^p w(x)dx.$$

(iv) The series $\sum_{i>1}\sum_j \Theta_{i,j} + \Theta_1 = \Theta$ converges in E^q_N, and for almost every x in $(x_{-\infty}, \infty)$,

$$N^+_{q,\alpha}(\Theta; x) \leq \sum_{i>1,j} N^+_{q,\alpha}(\Theta_{i,j}; x) + N^+_{q,\alpha}(\Theta_1; x). \tag{29}$$

(v) Furthermore,

$$\int N^+_{q,\alpha}(\Theta, x)\ ^p w(x)dx \leq c\int_{\Omega} N^+_{q,\alpha}(F, x)\ ^p w(x)dx.$$

(vi) If $G = F - \Theta$, $N^+_{q,\alpha}(G, x) \leq ct$.

Proof. The proof follows the lines of the argument in Lemma 10 of Gatto-Jiménez-Segovia [3]. Let us prove (i). First, we consider $i > 1$, and $j > 1$ and $x \in \hat{I}_{i,j}$. We can assume that $N^+_{q,\alpha}(F; x) < \infty$, otherwise, there is nothing to prove. Let $P(x, y)$ be the polynomial of degree at most N

satisfying $n_{q,a}(f(y)-P(x,y);x) = N^+_{q,\alpha}(F;x)$. Since $\mathrm{supp}(\eta_{i,j}) \subset I_{i,j}$, we have for $j>1$

$$\theta_{i,j}(y) = \eta_{i,j}(y)\,(f(y) - P(x_{i,j},y))$$

We define the polynomial

$$Q_{i,j}(x,y) = \sum_{k=0}^{N} D^k_y[\eta_{i,j}(y)(P(x,y) - P(x_{i,j},y))]\Big|_{y=x} \frac{(y-x)^k}{k!}.$$

Let us estimate $\rho^{-\alpha}\left|\theta_{i,j}(.) - Q_{i,j}(x,.)\right|_{q,[x,x+\rho]}$. We have that

$$Q_{i,j}(x,y) \tag{30}$$

$$= \sum_{k=0}^{N}\left[D^k_y(P(x,y) - P(x_{i,j},y))\Big|_{y=x} \frac{(y-x)^k}{k!} \times \left(\sum_{h=0}^{N-k} D^h\eta_{i,j}(x)\frac{(y-x)^h}{h!}\right)\right]$$

Let $r_{i,j} = \frac{b_i - a_i}{2^j}$ and we consider $y \in [x, x+\rho]$. Then, taking into account that $x_{i,j} = b_i$ if $j=2$ and $x_{i,j} = a_i$ if $j>2$, and by (2) of Lemma 4.1

$$|x_{i,j} - x| \leq 4r_{i,j}, \text{ and}$$

$$|y-x| + |y - x_{i,j}| \leq 2\,|y-x| + |x_{i,j} - x| \leq 4(\rho + r_{i,j})$$

Since $N^+_{q,\alpha}(F, x_{i,j}) \leq t < N^+_{q,\alpha}(F;x)$ and by Lemma 3.3, we get that

$$\left|D^k_y(P(x,y) - P(x_{i,j},y))\right| \leq C\; N^+_{q,\alpha}(F;x)\,(|y-x| + |x_{i,j} - x|)^{\alpha-k} \tag{31}$$

Assume first that $\rho \geq r_{i,j}$. In this case, we have

$$\begin{aligned} |\theta_{i,j}(y) - Q_{i,j}(x,y)| &= |\eta_{i,j}(y)\,(f(y) - P(x_{i,j},y)) - Q_{i,j}(x,y)| \\ &\leq \eta_{i,j}(y)\,|f(y) - P(x,y)| + \eta_{i,j}(y)\,|(P(x,y) - P(x_{i,j},y)| + |Q_{i,j}(x,y)| \end{aligned} \tag{32}$$

For the second term of the right hand side of this inequality, we obtain

$$\eta_{i,j}(y)\,|(P(x,y) - P(x_{i,j},y)| \leq c\; N^+_{q,\alpha}(F;x)\rho^\alpha. \tag{33}$$

Now, let us estimate $|Q_{i,j}(x,y)|$. From (31) with $y=x$, we have

$$\left| D^k_y(P(x,y) - P(x_{i,j},y))\Big|_{y=x}\right| \leq cN^+_{q,\alpha}(F;x)r^{\alpha-k}_{i,j}.$$

Then, from (30), by condition (4) of Lemma 4.1, and recalling that $\rho \geq r_{i,j}$ and $\alpha = N + \beta$ it follows that

$$\begin{aligned} |Q_{i,j}(x,y)| &\leq c\sum_{k=0}^{N} N^+_{q,\alpha}(F;x)r^{\alpha-k}_{i,j}\rho^k\left(\sum_{h=0}^{N-k} cr^{-h}_{i,j}\rho^h\right) \\ &\leq cN^+_{q,\alpha}(F;x)\,\rho^\alpha. \end{aligned} \tag{34}$$

Integrating (32) over $[x,x+\rho]$ and using the estimates (33) and (34), we get for $\rho \geq r_{i,j}$

$$\begin{aligned} &\rho^{-\alpha}\left|\theta_{i,j}(.) - Q_{i,j}(x,.)\right|_{q,[x,x+\rho]} \\ &\leq \rho^{-\alpha}\left|f(y) - P(x,y)\right|_{q,[x,x+\rho]} + c\; N^+_{q,\alpha}(F;x) \end{aligned} \tag{35}$$

Now we consider the case $\rho < r_{i,j}$. We rewrite $Q_{i,j}(x,y)$ as

$$Q_{i,j}(x,y)$$

$$= \sum_{k=0}^{N}\left[D^k\eta_{i,j}(x)\frac{(y-x)^k}{k!} \times \left(\sum_{h=0}^{N-k} D^h_y(P(x,y) - P(x_{i,j},y))\Big|_{y=x}\frac{(y-x)^h}{h!}\right)\right].$$

Adding and subtracting the expression

$$\eta_{i,j}(y)P(x,y) + \sum_{k=0}^{N} D^k\eta_{i,j}(x)\frac{(y-x)^k}{k!}(P(x,y) - P(x_{i,j},y))$$

to $Q_{i,j}(x,y)$ we obtain

$$\begin{aligned}
&|\theta_{i,j}(y) - Q_{i,j}(x,y)| \leq \eta_{i,j}(y)\,|f(y) - P(x,y)| \\
&+ \left| \left[\eta_{i,j}(y) - \sum_{k=0}^{N} D^k\eta_{i,j}(x)\frac{(y-x)^k}{k!} \right] \right| |(P(x,y) - P(x_{i,j},y)| \\
&+ \left| \sum_{k=0}^{N} \left[D^k\eta_{i,j}(x)\frac{(y-x)^k}{k!} \right] \times \right. \\
&\left. \left[(P(x,y) - P(x_{i,j},y) - \left(\sum_{h=0}^{N-k} D^h_y(P(x,y) - P(x_{i,j},y))\Big|_{y=x} \frac{(y-x)^h}{h!} \right) \right] \right| \\
&\leq |f(y) - P(x,y)| + S_1 + S_2.
\end{aligned} \tag{36}$$

If $y \in [x, x+\rho]$, and considering condition (4) of Lemma 4.1 and (31), recalling that $\alpha = N + \beta$ and $\rho < r_{i,j}$, we get

$$\begin{aligned}
S_1 &= \left| D^{N+1}\eta_{i,j}(\xi)\frac{(y-x)^{N+1}}{N+1!} \right| |(P(x,y) - P(x_{i,j},y)| \\
&\leq c\ r_{i,j}^{-(N+1)}\rho^{N+1}N^+_{q,\alpha}(F;x)\ r^{\alpha}_{i,j} \leq cN^+_{q,\alpha}(F;x)\rho^{\alpha}
\end{aligned}$$

As for S_2, similar arguments show that

$$\begin{aligned}
S_2 &= \left| \sum_{k=1}^{N} \left[D^k\eta_{i,j}(x)\frac{(y-x)^k}{k!} \right] \left[D^{N+1-k}_y(P(x,y) - P(x_{i,j},y))\Big|_{y=\xi} \frac{(y-x)^{N+1-k}}{k!} \right] \right| \\
&\leq c\ \sum_{k=1}^{N} r_{i,j}^{-k}\rho^k N^+_{q,\alpha}(F;x)\ (r_{i,j} + \rho)^{k+\beta-1}\rho^{N+1-k} \leq cN^+_{q,\alpha}(F;x)\rho^{\alpha}.
\end{aligned}$$

Integrating (36) and by the estimates just obtained we get that (35) also holds for $\rho < r_{i,j}$. This shows that

$$N^+_{q,\alpha}(\Theta_{i,j},x) \leq c\ N^+_{q,\alpha}(F;x).$$

Now we consider the classes $\Theta_{i,1}$ with $i > 1$ and $x \in \hat{I}_{i,1} = (a_i + 2^{-2}(b_i - a_i), b_i)$. We can express $\theta_{i,1}(y)$ as following way

$$\begin{aligned}
\theta_{i,1}(y) &= \eta_{i,1}(y)\chi_{I_i}(y)\,(f(y) - P(b_i,y)) \\
&= \eta_{i,1}(y)\,(f(y) - P(b_i,y)) - \eta_{i,1}(y)\chi_{[b,\infty)}(y)\,(f(y) - P(b_i,y)) \\
&= \theta^1_{i,1}(y) - \theta^2_{i,1}(y).
\end{aligned} \tag{37}$$

We denote $\Theta^1_{i,1}$ and $\Theta^2_{i,1}$ the classes of $\theta^1_{i,1}(y)$ and $\theta^2_{i,1}(y)$ respectively.

For the class $\Theta^1_{i,1}$, arguing as before we get

$$N^+_{q,\alpha}(\Theta^1_{i,1},x) \leq cN^+_{q,\alpha}(F,x),$$

for all $x \in \hat{I}_{i,1}$. Now we consider the class $\Theta^2_{i,1}$. Let us estimate $\int_x^{x+\rho} \left|\theta^2_{i,1}(y)\right|^q dy$. Since $\mathrm{supp}(\theta^2_{i,1}) \subset [b_i, a_i + 2(b_i - a_i)]$, we can assume that $x + \rho > b_i$, if not the integral that we want to estimate is equal to zero. By (1) of Lemma 4.1, and since $x \leq b_i$, we have

$$\int_x^{x+\rho} \left|\theta_{i,1}^2(y)\right|^q dy \leq \int_{b_i}^{b_i+\rho} |f(y) - P(b_i, y)|^q \, dy \leq N_{q,\alpha}^+(F, b_i)^q \, \rho^{\alpha q+1},$$

and since $N_{q,\alpha}^+(F, b_i) \leq t < N_{q,\alpha}^+(F; x)$ we obtain $N_{q,\alpha}^+(\Theta_{i,1}^2, x) \leq N_{q,\alpha}^+(F; x)$. Then, it follows that

$$N_{q,\alpha}^+(\Theta_{i,1}, x) \leq c N_{q,\alpha}^+(F; x).$$

To finish the proof of (i) let us estimate $N_{q,\alpha}^+(\Theta_1, x)$. Let $x \in I_1 = (x_{-\infty}, b_1)$. We define the polynomial $Q_1(x, y) = P(x, y) - P(b_1, y)$. Let us estimate $\rho^{-\alpha} \left|\theta_1(.) - Q_1(x, .)\right|_{q,[x,x+\rho]}$. We assume first that $x + \rho \leq b_1$. In this case, by (15) we have

$$\begin{aligned}\rho^{-\alpha} \left|\theta_1(.) - Q_1(x, .)\right|_{q,[x,x+\rho]} &= \frac{1}{\rho^{\alpha+1/q}} \left(\int_x^{x+\rho} |f(y) - P(x, y)|^q \, dy\right)^{1/q} \\ &\leq N_{q,\alpha}^+(F; x).\end{aligned}$$

If $x + \rho > b_1$, using Lemma 3.3 we obtain that

$$\begin{aligned}\rho^{-\alpha} \left|\theta_1(.) - Q_1(x, .)\right|_{q,[x,x+\rho]} &\leq N_{q,\alpha}^+(F; x) + \rho^{-\alpha-1/q} \left(\int_{b_1}^{x+\rho} |P(x, y) - P(b_1, y)|^q \, dy\right)^{1/q} \\ &\leq c(N_{q,\alpha}^+(F; x) + N_{q,\alpha}^+(F, b_1)) \leq c N_{q,\alpha}^+(F; x),\end{aligned}$$

Then, $N_{q,\alpha}^+(\Theta_1, x) \leq c N_{q,\alpha}^+(F; x)\chi_{I_1}(x)$ if $x \in I_1$. Moreover, since $\theta_1(y) = 0$ if $y > b_1$, we have that $N_{q,\alpha}^+(\Theta_1, x) = 0$ if $x \geq b_1$.

Let us prove condition (ii). Again, we work first with the classes $\Theta_{i,j}$ for $i > 1$ and $j > 1$. Let $x > x_{-\infty}$, and $x \notin \hat{I}_{i,j}$. We will estimate $\rho^{-\alpha} \left|\theta_{i,j}\right|_{q,[x,x+\rho]}$. If $x_{-\infty} < x < a_i + 2^{-j-1}(b_i - a_i)$, since $\operatorname{supp}(\theta_{i,j}) \subset I_{i,j}$, we have that $\left|\theta_{i,j}\right|_{q,[x,x+\rho]}$ is equal to zero unless $[x, x + \rho] \cap I_{i,j} \neq \emptyset$. Then,

$$\rho > a_i + 2^{-j}(b_i - a_i) - x. \tag{38}$$

On the other hand, since $\operatorname{supp}(\eta_{i,j}) \subset I_{i,j} \subset [a_i, a_i + 4r_{i,j}]$, we get,

$$\rho^{-\alpha} \left|\theta_{i,j}\right|_{q,[x,x+\rho]} = \rho^{-\alpha-1/q} \left(\int_x^{x+\rho} \eta_{i,j}(y)^q \, |f(y) - P(x_{i,j}, y)|^q \, dy\right)^{1/q} \tag{39}$$

$$\leq \frac{1}{\rho^\alpha} \left|f(.) - P(a_i, .)\right|_{q,[a_i,a_i+4r_{i,j}]} + \frac{1}{\rho^{\alpha+1/q}} \left(\int_{a_i}^{a_i+4r_{i,j}} |P(x_{i,j}, y) - P(a_i, y)|^q \, dy\right)^{1/q}.$$

Since $x_{i,j} = a_i$ for $j > 2$, the second summand of the last line is null except in the case $j = 2$. In this case $x_{i,2} = b_i$ and using Lemma 3.3, we obtain

$$\left(\int_{a_i}^{a_i+4r_{i,j}} |P(x_{i,j}, y) - P(a_i, y)|^q \, dy\right)^{1/q} \leq c \, t \, r_{i,j}^{\alpha+1/q}.$$

Therefore, substituting in (39) and by (38), we get

$$\rho^{-\alpha} \left|\theta_{i,j}\right|_{q,[x,x+\rho]} \leq ct \left(\frac{r_{i,j}}{a_i + 2^{-j}(b_i - a_i) - x}\right)^{\alpha+1/q} \leq ct \, \left[M^+\chi_{\hat{I}_{i,j}}(x)\right]^{\alpha+1/q}.$$

This implies that

$$N_{q,\alpha}^+(\Theta_{i,j}; x) \leq ct \, \left[M^+\chi_{\hat{I}_{i,j}}(x)\right]^{\alpha+1/q}, \tag{40}$$

for $x_{-\infty} < x \leq a_i + 2^{-j-1}(b_i - a_i)$. Since $\theta_{i,j}(y)$ is equal to zero for $y > a_i + 2^{-j+2}(b_i - a_i)$, we have that (40) also holds for $x > a_i + 2^{-j+2}(b_i - a_i)$. By (37) and using a similar argument we obtain (ii) for the classes $\Theta_{i,1}$.

As for condition (iii). Since $w \in A^+_{(\alpha+1/q)p}$ and $(\alpha + 1/q)\,p > 1$ and taking into account (i), (ii) and (2) of Lemma 3.1, we have

$$\int_{x_{-\infty}}^{\infty} \left(\sum_{i=2}^{\infty} \sum_{j=1}^{\infty} N^+_{q,\alpha}(\Theta_{i,j}; x) + N^+_{q,\alpha}(\Theta_1; x) \right)^p w(x)dx$$

$$\leq c \int_{I_1} N^+_{q,\alpha}(F; x)^p w(x)dx + c \sum_{i=2}^{\infty} \sum_{j=1}^{\infty} \int_{\hat{I}_{i,j}} N^+_{q,\alpha}(F; x)^p w(x)dx$$

$$+ \sum_{i=2}^{\infty} \sum_{j=1}^{\infty} ct^p \int_{(x_{-\infty},\infty)} \left[M^+ \chi_{\hat{I}_{i,1}}(x) \right]^{(\alpha+1/q)p} w(x)dx$$

$$\leq c \int_{\Omega} N^+_{q,\alpha}(F; x)^p w(x)dx + ct^p w(\Omega) \leq c \int_{\Omega} N^+_{q,\alpha}(F; x)^p w(x)dx.$$

Condition (iv) is a consequence of condition (iii) and Lemma 3.6. As for condition (v), it follows from conditions (iii) and (iv).

Now we will prove (vi). We consider a point $x_0 \notin \Omega$, such that

$$\sum_{i>1,j} N^+_{q,\alpha}(\Theta_{i,j}; x_0) + N^+_{q,\alpha}(\Theta_1; x_0) < \infty.$$

Since $\theta_{i,j}(y)$ and $\theta_1(y)$ are the representatives satisfying $N^+_{q,\alpha}(\Theta_{i,j}; x_0) = n^+_{q,\alpha}(\theta_{i,j}; x_0)$ and $N^+_{q,\alpha}(\Theta_1; x_0) = n^+_{q,\alpha}(\theta_1; x_0)$, by Lemma 3.6,

$$\theta(y) = \sum_{i=2}^{\infty} \sum_{j=1}^{\infty} \eta_{i,j}(y) \chi_{I_i}(y) \left(f(y) - P(x_{i,j}, y) \right) + \chi_{I_1}(y)(f(y) - P(b_1, y))$$

is a representative of Θ and therefore

$$g(y) = \begin{cases} \sum_{i=2}^{\infty} \sum_{j=1}^{\infty} \eta_{i,j}(y) \chi_{I_i}(y) P(x_{i,j}, y) + \chi_{I_1}(y) P(b_1, y)) \text{ if } x \in \Omega, \\ f(y) \text{ if } x \notin \Omega, \end{cases}$$

is a representative of $G = F - \Theta$. Thus, by Lemma 4.2 $N^+_{q,\alpha}(G; x) \leq ct$. ■

Proof of Theorem 2.1. The method that we will use to prove the theorem it was developed in [4]. Proceedings as in [4] we can show, as a consequence of Lemma 4.3, that if H is an element of E^q_N satisfying $N^+_{q,\alpha}(H; x) \leq 1$ and $\int N^+_{q,\alpha}(H; x)^r w(x)dx < \infty$, for some $0 < r < p \leq 1$, $(\alpha + 1/q)r > 1$ and such that $w \in A^+_{(\alpha+1/q)r}$ then there exists a numerical sequence $\{\lambda_i\}$ and a sequence of p-atoms $\{A_i\}$ of $\mathcal{H}^{p,+}_{q,\alpha}(w)$ such that $H = \sum \lambda_i A_i$ in $\mathcal{H}^{p,+}_{q,\alpha}(w)$. Moreover, $\sum |\lambda_i|^p \leq \int N^+_{q,\alpha}(H; x)^r w(x)dx$.

From this fact, the proof of theorem can be obtained following the same lines of the proof of the Theorem 4.3 of [4]. ■

In order to prove Corollary 2.2 we will need the following lemma.

Lemma 4.4. *Sea $I = (-\infty, b)$. There exists a sequence $\{\nu_j\}_{j=-\infty}^{\infty}$ of C_0^∞ functions satisfying the following conditions*

1) $0 \leq \nu_j(x) \leq 1$ and $\sum_j \nu_j(x) = \chi_{(-\infty,b)}(x)$.

2) For each integer j, if we denote $I_j = [-2^{-j} + b, -2^{-j-2} + b]$ then $\text{supp}(\nu_j) \subset I_j$. Let $r_j = \frac{1}{2^j}$, then for every $x \in I_j$, $r_j \leq b - x \leq cr_j$.
3) the number of interval I_j that intersect to other interval I_k does not exceed two.
4) If k is an integer, $k \geq 0$, we have

$$\left| D^k \nu_j(x) \right| \leq C_k r_j^{-k}$$

where C_k does not depend on j.

See [6], pag. 167

Lemma 4.5. *Given a p-atom A in $\mathcal{H}^{p,+}_{q,\alpha}(w)$ there exists a numerical sequence $\{\mu_k\}$, and a sequence of p-atoms $\{A_k\}$ in $\mathcal{H}^{p,+}_{q,\alpha}(w)$ with bounded associated intervals, such that*

$$A = \sum \mu_k A_k \ \textit{in } E^q_N \quad \textit{and} \sum |\mu_k|^p \leq C, \tag{41}$$

where C is a finite constant not depending of A.

Proof. If there exists a bounded interval associated to the p-atom A the result is immediate. Then, we assume that $w((-\infty, b)) < \infty$, where $(-\infty, b)$ is an interval associated to p-atom A. Let $a(y)$ be the representative of A, such that $\text{supp}(a) \subset I = (-\infty, b]$, and we denote $P(x, y)$ the polynomial of degree at most N, such that $N^+_{q,\alpha}(A; x) = N^+_{q,\alpha}(a(y) - P(x, y); x)$. We observe that $N^+_{q,\alpha}(A; b) = 0$ and $P(b, y) \equiv 0$. We consider the sequence of functions $\{\nu_j\}_{j=-\infty}^{\infty}$ of Lemma 4.4 associated to interval $I = (-\infty, b)$. Then, by condition (1) of Lemma 4.4

$$a(x) = \sum_{j=-\infty}^{\infty} \nu_j(x) a(x) = \sum_{j=-\infty}^{\infty} \theta_j(x) \tag{42}$$

For each integer j, we denote Θ_j the class in E^q_N of the function $\theta_j(x) = \nu_j(x)a(x)$. We claim that

$$N^+_{q,\alpha}(\Theta_j; x) \leq Cw(I)^{-1/p} \text{ for all } x, \tag{43}$$

where C does not depend of j. By (2) of Lemma 4.4, $\text{supp}(\nu_j(y)a(y)) \subset I_j = [-2^{-j} + b, -2^{-j-2} + b]$. Then, $N^+_{q,\alpha}(\Theta_j; x) = 0$ if $x \geq -2^{-j-2} + b$. Now, we suppose that $x \leq -2^{-j-2} + b$. For this case, since $P(b, y) \equiv 0$ and by Lemma 3.3 we have

$$\left| D^k P(x, y) \right| = \left| D^k \left[P(x, y) - P(b, y) \right] \right| \leq cw(I)^{-1/p} (|y - x| + |b - y|)^{\alpha - k},$$

Taking into account this estimate, the conditions of Lemma 4.4 and proceeding as in the proof of (i) in Lemma 4.3, we obtain (43) For each integer j, we define

$$\mu_j = C \left(\frac{w(I_j)}{w(I)} \right)^{1/p} \text{ and } a_j(y) = \mu_j^{-1} \theta_j(y),$$

where C is the constant in (43). We denote by A_j the class in E^q_N of $a_j(y)$. Then, by (43), we have $N^+_{q,\alpha}(A_j; x) \leq w(I_j)^{-1/p}$ and $\text{supp}(a_j) \subset I_j$. Then, the classes A_j are p-atom in $\mathcal{H}^{p,+}_{q,\alpha}(w)$ with bounded associated intervals. Using (3) of Lemma 4.4, we get $\sum_{j=-\infty}^{\infty} |\mu_j|^p \leq C$. It is not difficult to show that the norm in $\mathcal{H}^{p,+}_{q,\alpha}(w)$ of a p-atom is bounded by a constant C not depending of the p-atom, then we have that

$$\sum_{j=-\infty}^{\infty} \|\mu_j A_j\|^p_{\mathcal{H}^{p,+}_{q,\alpha}(w)} = \sum_{j=-\infty}^{\infty} |\mu_j|^p \|A_j\|^p_{\mathcal{H}^{p,+}_{q,\alpha}(w)} \leq C \sum_{j=-\infty}^{\infty} |\mu_j|^p < \infty.$$

Thus, by Corollary 3.7 there exists F in $\mathcal{H}^{p,+}_{q,\alpha}(w)$ such that $F = \sum_{j=-\infty}^{\infty} \mu_j A_j$ in $\mathcal{H}^{p,+}_{q,\alpha}(w)$ and by Corollary 3.5 $F = \sum_{j=-\infty}^{\infty} \mu_j A_j$ in E^q_N, and by (ii) of Lemma 3.6 and (42) we have that $F = A$ in E^q_N. ■

Proof of Corrolary 2.2. By Theorem 2.1, we have

$$F = \sum_k \lambda_k A_k \text{ in } E^q_N, \text{ and } \sum |\lambda_k|^p \leq c \|F\|^p_{\mathcal{H}^{p,+}_{q,\alpha}(w)}.$$

Now, applying Lemma 4.5, we can express each p-atom A_k as

$$A_k = \sum_j \mu_{k,j} A_{k,j} \text{ in } E^q_N, \text{ where } \sum_j |\mu_{k,j}|^p \leq C,$$

and the associated intervals to the p-atoms $A_{k,j}$ are bounded. Then,

$$F = \sum_k \sum_j \lambda_k \mu_{k,j} A_{k,j} \text{ in } E^q_N, \text{ and}$$

$$\sum_k \sum_j |\lambda_k \mu_{k,j}|^p \leq \sum_k |\lambda_k|^p \sum_j |\mu_{k,j}|^p \leq C \sum_k |\lambda_k|^p \leq C \|F\|^p_{\mathcal{H}^{p,+}_{q,\alpha}(w)}.$$

■

Lemma 4.6. *Let $f \in \mathcal{D}'(x_{-\infty}, \infty)$, and we suppose that $D^{N+1} f \equiv 0$. Then f agrees with a polynomial of degree less than or equal to $N+1$ in $\mathcal{D}'(x_{-\infty}, \infty)$.*

This is well known and we will be omitted its proof.

The following lemma proves the first part of Theorem 2.3.

Lemma 4.7. *Let $w \in A^+_s$ and $(\alpha + 1/q)\, p \geq s > 1$ or $(\alpha + 1/q)\, p > 1$ if $s = 1$, where $0 < p \leq 1$, and let $\gamma \geq \alpha$. If $a(y)$ is a p-atom in $H^p_{+,\gamma}(w)$ then*

$$\|\overline{P_\alpha a}\|_{\mathcal{H}^{p,+}_{q,\alpha}(w)} \leq C,$$

where C is a finite constant not depending on $a(y)$.

Proof. Without loss of generality, we can suppose that $x_{-\infty} \leq 0$ and $\mathrm{supp}(a) \subset I = [0, b]$. Let $x \in (x_{-\infty}, \infty)$ and $z > 0$. As in(11) of Lemma 3.11, we define

$$R(x, z) =$$

$$\frac{1}{\Gamma(\alpha)}[\int_{x+z}^{\infty} (y - x - z)^{\alpha - 1} a(y) dy - \int_x^{\infty} (\sum_{k=0}^{N} c_{k,\alpha} (y - x)^{\alpha - 1 - k} z^k) a(y) dy\, z^k].$$

We suppose that $x < -4b$. We observe that if there exists $x \in (x_{-\infty}, \infty)$ such that $x < -4b$ then $d(x_{-\infty}, I) > |I| = b$, therefore $a(y)$ has vanishing moments up to order $\gamma - 1$ and since $\gamma \geq \alpha = N + \beta$, we have that $a(y)$ has vanishing moments up to order N. We will prove the followings estimates

(i) If $z \le \frac{|x|}{2}$,

$$|R(x,z)| \le Cw(I)^{-1/p}\left(\frac{b}{|x|}\right)^{\alpha+1} z^{\alpha}.$$

(ii) If $z > \frac{|x|}{2}$ and $|x+z| > 2b$ then

$$|R(x,z)| \le Cw(I)^{-1/p}\frac{b^{N+2}}{|x+z|^{2-\beta}} + Cw(I)^{-1/p}\left(\frac{b}{|x|}\right)^{\alpha+1} z^{\alpha}.$$

(iii) If $z > \frac{|x|}{2}$ and $|x+z| \le 2b$ then

$$|R(x,z)| \le Cw(I)^{-1/p}\left(\frac{b}{|x|}\right)^{\alpha} z^{\alpha}.$$

Let us consider (i) . We get that

$$\begin{aligned} R(x,z) &= \frac{1}{\Gamma(\alpha)}\int_{x+z}^{\infty}[(y-x-z)^{\alpha-1} - \sum_{k=0}^{N} c_{k,\alpha}\,(y-x)^{\alpha-1-k}\, z^k a(y)]dy \\ &\quad -\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{N} c_{k,\alpha}\int_{x}^{x+z}(y-x)^{\alpha-1-k}\,a(y)dy\; z^k = R_1 + R_2. \end{aligned}$$

Thus since $x+z \le 0$ and $\mathrm{supp}(a) \subset [0,b]$, it follows that R_2 vanishes. As for R_1, by Taylor's formula and Lemma 3.12, we have that

$$\begin{aligned} |R_1| &\le \left|D^{N+1}P_{\alpha}a(x-\theta z)\right| z^{N+1} \le Cw(I)^{-1/p}\frac{b^{N+2}}{|x|^{2+N+1-\beta}} z^{N+1} \\ &\le Cw(I)^{-1/p}\left(\frac{b}{|x|}\right)^{\alpha+1}\left(\frac{z}{|x|}\right)^{N+1-\alpha} z^{\alpha} \le Cw(I)^{-1/p}\left(\frac{b}{|x|}\right)^{\alpha+1} z^{\alpha}, \end{aligned}$$

which implies (i).

We observe that

$$|R(x,z)| \le |P_{\alpha}a(x+z)| + C\sum_{k=0}^{N}\left|D^k P_{\alpha}a(x)\right| z^k \tag{44}$$

Then, if $z > \frac{|x|}{2}$, by Lemma 3.12, we obtain

$$\left|D^k P_{\alpha}a(x)\right| z^k \le cw(I)^{-1/p}\frac{b^{N+2}z^k}{|x|^{2+k-\beta}} \le cw(I)^{-1/p}\left(\frac{b}{|x|}\right)^{\alpha+1} z^{\alpha}. \tag{45}$$

In case (ii), i.e., when $|x+z| > 2b$, applying Lemma 3.12 with $k=0$, we get

$$|P_{\alpha}a(x+z)| \le Cw(I)^{-1/p}\frac{b^{N+2}}{|x+z|^{2-\beta}},$$

and thus (ii) holds. As for (iii), we have that $|x+z| \le 2b$, then it follows that

$$|P_{\alpha}a(x+z)| \le C\,\|a\|_{\infty}\int_{x+z}^{b}|y-x-z|^{\alpha-1}\,dy \le Cw(I)^{-1/p}b^{\alpha},$$

therefore, since $\frac{z}{|x|} > 1/2$, we obtain

$$|P_{\alpha}a(x+z)| \le Cw(I)^{-1/p}\left(\frac{b}{|x|}\right)^{\alpha} z^{\alpha}.$$

Then, from (44), the estimate above and (45), we get (iii).

Taking into account (i), (ii) and (iii) and arguing as the proof of Theorem 1 in [3] we obtain that for $x < -4b$

(46)
$$N^{+}_{q,\alpha}(\overline{P_\alpha a};x) \leq Cw(I)^{-1/p}\left(\frac{b}{|x|}\right)^{\alpha+1/q} \leq Cw(I)^{-1/p}\left(M^{+}\chi_I(x)\right)^{\alpha+1/q}$$

holds. This estimate also holds if $x > b$, since $P_\alpha a(x) = 0$ for $x > b$. If $-4b \leq x < b$, by Lemma 3.11, and since $\|a\|_\infty \leq w(I)^{-1/p}$, we get

$$N^{+}_{q,\alpha}(\overline{P_\alpha a};x) \leq Cw(I)^{-1/p}. \tag{47}$$

Since $M^{+}\chi_I(x) \geq 1/5$ if $x \in [-4b, b)$, it follows that (46) holds for every $x \in (x_{-\infty}, \infty)$. Then, the lemma follows from (46) and part (2) of Lemma 3.1. If $d(x_{-\infty}, I) \leq |I|$ the conclusion of lemma follows from (47) and part (3) of Lemma 3.1. ■

Proof of Theorem 2.3. The first part of theorem follows from Theorem 1.2 and Lemma 4.7.

Now, we suppose that α is a natural number. Then, if $a(y)$ is a p-atom in $H^{p}_{+,\gamma}(w)$, it is not difficult to see that

$$D^\alpha P_\alpha a(x) = (-1)^\alpha\, a(x). \tag{48}$$

We will study the application D^α in $\mathcal{H}^{p,+}_{q,\alpha}(w)$. Let $F \in \mathcal{H}^{p,+}_{q,\alpha}(w)$. Since $N^{+}_{q,\alpha}(F;x) \in L^p(w)$, $N^{+}_{q,\alpha}(F;x)$ is finite almost every point $x \in (x_{-\infty}, \infty)$; we consider a point x in this conditions and Let f be the representative of F satisfying $N^{+}_{q,\alpha}(F;x) = n^{+}_{q,\alpha}(f;x)$. Let $\phi \in \Phi_\gamma(x)$ and we suppose that $\mathrm{supp}(\phi) \subset I_\phi = [x, c]$. Then, by the definition of $D^\alpha F$, taking into account that $\alpha \leq \gamma$ (then $\phi \in \Phi_\alpha(x)$) and applying the Hölder's inequality, we obtain

$$\begin{aligned}
|\langle D^\alpha F, \phi\rangle| &= |\langle D^\alpha f, \phi\rangle| = |\langle f, D^\alpha \phi\rangle| = \left|\int_{I_\phi} f(y) D^\alpha\phi(y)dy\right| \\
&= \left|\int_x^c f(y)D^\alpha\phi(y)dy\right| \leq \frac{1}{|I_\phi|^{\alpha+1}}\int_x^c |f(y)|\,dy \\
&\leq \frac{1}{|I_\phi|^\alpha}\left(\frac{1}{|I_\phi|}\int_x^c |f(y)|^q\,dy\right)^{1/q} \leq N^{+}_{q,\alpha}(F;x).
\end{aligned}$$

Therefore $(D^\alpha F)^{*}_{+,\gamma}(x) \leq N^{+}_{q,\alpha}(F;x)$, which implies that

$$\|D^\alpha F\|_{H^p_{+,\gamma}(w)} \leq \|F\|_{\mathcal{H}^{p,+}_{q,\alpha}(w)}. \tag{49}$$

We denote $\widetilde{P}_\alpha f$ the extension of the first part of Theorem. We will prove that $\widetilde{P}_\alpha$ is onto. Let $F \in \mathcal{H}^{p,+}_{q,\alpha}(w)$, by (49) $D^\alpha F \in H^p_{+,\gamma}(w)$, then by Theorem 1.2 we have that

$$D^\alpha F = \sum_j \lambda_j a_j, \text{ where } \sum_j |\lambda_j|^p \sim \|D^\alpha F\|^p_{H^p_{+,\gamma}(w)}. \tag{50}$$

Then, if we denote $f = (-1)^\alpha \sum_j \lambda_j a_j$, that belongs to $H^p_{+,\gamma}(w)$ we get that

$$\widetilde{P}_\alpha f = (-1)^\alpha \sum_j \lambda_j P_\alpha a_j \in \mathcal{H}^{p,+}_{q,\alpha}(w). \tag{51}$$

As consequence of Lemma 4.6, we have that D^α is one to one. From this fact, (51) and (48) we obtain that $\widetilde{P}_\alpha f = F$. The fact that $\widetilde{P}_\alpha$ is one to one is consequence of Theorem 1.2, (49) and (48) ■

We observe that the last Theorem, its proof and Theorem 1.2 give other proof of the Theorem 2.1, always that α is a natural number.

To finish, we will observe that in general, in the case that α is not a natural number, the extension $\widetilde{P}_\alpha$ is not onto. We suppose that $0 < \alpha < 1$, $w \equiv 1$, and $(\alpha + 1/q)p > 1$. Let $\phi \in C_0^\infty$, and we assume that $|\phi| \geq c > 0$ in some interval. We define

$$a(x) = \phi(x) \left(\sum_{n=1}^{\infty} \frac{\cos 2^n \pi x}{2^{n\alpha}} \right).$$

It is well know that the previous series defines a Lipschitz-α function (e.g. see [9]), then $a(x)$ is a Lipschitz-α function. If we denote by A the class of $a(x)$ in E_0^q, we have that $N_{q,\alpha}^{+}(A;x)$ is bounded and since $\text{supp}(a(y)) \subset I$ for some interval I, we get that $A \in \mathcal{H}_{q,\alpha}^{p,+}(1)$. However, It can be shown that does not exist any distribution f in $H_{+,\gamma}^p(1)$ such that $A = \widetilde{P}_\alpha f$.

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