

BOUNDEDNESS OF THE WEYL FRACTIONAL INTEGRAL ON ONE-SIDED WEIGHTED LEBESGUE AND LIPSCHITZ SPACES

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ABSTRACT. In this paper we introduce the one-sided weighted spaces $\mathcal{L}_w^-(\beta)$, $-1 < \beta < 1$. The purpose of this definition is to obtain an extension of the Weyl fractional integral operator I_α^+ from L_w^p into a suitable weighted space.

Under certain condition on the weight w , we have that $\mathcal{L}_w^-(0)$ coincides with the dual of the Hardy space $H_-^1(w)$. We prove for $0 < \beta < 1$, that $\mathcal{L}_w^-(\beta)$ consists of all functions satisfying a weighted Lipschitz condition. In order to give another characterization of $\mathcal{L}_w^-(\beta)$, $0 \leq \beta < 1$, we also prove a one-sided version of John-Nirenberg Inequality.

Finally, we obtain necessary and sufficient conditions on the weight w for the boundedness of I_α^+ from L_w^p into $\mathcal{L}_w^-(\beta)$, $-1 < \beta < 1$, and its extension to a bounded operator from $\mathcal{L}_w^-(0)$ into $\mathcal{L}_w^-(\alpha)$.

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1. NOTATIONS, DEFINITIONS AND PREREQUISITES

Let $E \subset \mathbb{R}$ be a Lebesgue measurable set. We shall denote its Lebesgue measure by $|E|$ and the characteristic function of E by χ_E .

As usual, a weight w is a measurable, non-negative and locally integrable function defined on $\mathbb{R}$.

Let w be a weight. Given a Lebesgue measurable set $E \subset \mathbb{R}$, its w -measure will be denote by $w(E) = \int_E w(t)dt$.

Let $1 < p < \infty$. The weight w belongs to the class A_p^- if there exists a constant C such that

$$\sup_{h>0} \left[\frac{1}{h^p} \int_a^{a+h} w(x)dx \left(\int_{a-h}^a w(x)^{-\frac{1}{p-1}} dx \right)^{p-1} \right] \leq C,$$

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for all real number a . In a similar way, w belongs to A_p^+ if

$$\sup_{h>0} \left[\frac{1}{h^p} \int_{a-h}^a w(x) dx \left(\int_a^{a+h} w(x)^{-\frac{1}{p-1}} dx \right)^{p-1} \right] \leq C,$$

for all real number a . The class A_1^- is defined by the condition

$$\sup_{h>0} \left[\frac{1}{h} \int_a^{a+h} w(x) dx \right] \leq Cw(a),$$

for almost every real number a . The weight w belongs to A_1^+ if

$$\sup_{h>0} \left[\frac{1}{h} \int_{a-h}^a w(x) dx \right] \leq Cw(a),$$

for almost every a . These classes A_p^- and A_p^+ were introduced by E. Sawyer in [12]. We recall three basic results on these weights.

(i) For $1 < p < \infty$, a weight w belongs to A_p^- if and only if $w^{1-p'}$ belongs to $A_{p'}^+$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

(ii) If $1 \leq p < q < \infty$, then $A_p^- \subset A_q^-$.

(iii) If $1 < p < \infty$ and w belongs to A_p^- , then w belongs to $A_{p-\epsilon}^-$ for some $\epsilon > 0$.

The proof of parts (i) and (ii) are very simple and (iii) can be found in Proposition 3 in [3].

In the sequel, for each bounded interval $I = [a, b]$ we shall denote $I^- = [a - |I|, a]$ and $I^+ = [b, b + |I|]$.

Let $1 \leq q < \infty$. A weight w satisfies the condition $RH^-(q)$ if there exists a constant C such that for every bounded interval I ,

$$\left[\frac{1}{|I|} \int_I w(x)^q dx \right]^{1/q} \leq C \frac{1}{|I|} \int_{I^-} w(x) dx.$$

We shall say that a weight w belongs to D^- if there exists a constant C such that for every bounded interval I ,

$$w(I \cup I^+) \leq Cw(I).$$

It is well known that if $w \in A_p^-$, $1 \leq p < \infty$, then $w \in D^-$.

Let w be a weight, $1 \leq p < \infty$ and f a measurable function. We shall say that f belongs to L_w^p if

$$\|f\|_{p,w}^p = \int_{-\infty}^{\infty} \left[\frac{|f(x)|}{w(x)} \right]^p dx$$

is finite. The function f belongs to $\widetilde{L}_w^p$ if

$$[f]_{p,w}^p = \sup_{t>0} t^p \left| \left\{ x \in \mathbb{R} : \frac{|f(x)|}{w(x)} > t \right\} \right|$$

is finite.

Let $0 < \alpha < 1$. Given f a measurable function on $\mathbb{R}$, its Weyl fractional integral is defined by

$$I_\alpha^+ f(x) = \int_x^\infty \frac{f(y)}{(y-x)^{1-\alpha}} dy,$$

whenever this integral is finite.

In the sequel, the letter C will denote a positive finite constant not necessarily the same at each occurrence. If $1 \leq p \leq \infty$ then p' will be its conjugate exponent, that is, $1/p + 1/p' = 1$.

Let w be a weight and $-1 < \beta < 1$.

Definition 1.1. *We say that a locally integrable function f defined on $\mathbb{R}$ belongs to $\mathcal{L}_w(\beta)$, if there exists a constant C such that*

$$\frac{1}{w(I)|I|^\beta} \int_I |f(y) - f_I| dy \leq C,$$

for every bounded interval I , where $f_I = \frac{1}{|I|} \int_I f$. The least constant C will be denoted $\|f\|_{\mathcal{L}_w(\beta)}$.

The spaces $\mathcal{L}_w(\beta)$ were introduced by E. Harboure, O. Salinas and B. Viviani in [1]. They are a weighted version of the spaces $\mathcal{L}_{\lambda,p}$, for $p = 1$, defined by J. Peetre in [8]. If w belongs to A_q^- , $1 \leq q < 2$, then $\mathcal{L}_w(0)$ is the dual space of the one-sided weighted Hardy space $H_-^1(w)$, see [10] and [11].

Definition 1.2. *We say that a locally integrable function f defined on $\mathbb{R}$ belongs to $\mathcal{L}_w^-(\beta)$, if there exists a constant C such that*

$$\frac{1}{w(I^-)|I|^\beta} \int_I |f(y) - f_I| dy \leq C,$$

for every bounded interval I . The least constant C satisfying this inequality will be denoted $\|f\|_{\mathcal{L}_w^-(\beta)}$.

In the following definition, we consider a one-sided version of the classes $H(\alpha, p)$ defined in [1].

Definition 1.3. *Let $0 < \alpha < 1$ and $1 < p \leq \infty$. We say that a weight w belongs to $H^-(\alpha, p)$ if there exists a constant C such that for every bounded interval $I = [a, b]$, the inequality*

$$|I|^{\frac{1}{p}-\alpha+1} \left[\int_b^\infty \frac{w(y)^{p'}}{(y-a)^{(2-\alpha)p'}} dy \right]^{1/p'} \leq C \frac{w(I)}{|I|},$$

holds.

2. STATEMENT OF THE MAIN RESULTS

Lemma 4.1 part (iii) shows that if w belongs to $H^-(\alpha, p)$, $1 < p \leq \infty$, then w belongs to D^- and therefore $\mathcal{L}_w(\beta) \subseteq \mathcal{L}_w^-(\beta)$ for every $\beta : -1 < \beta < 1$. The next theorem states that w belonging to D^- is a sufficient condition for the equality of these spaces, whenever $0 \leq \beta < 1$.

Theorem 2.1. *Let $0 \leq \beta < 1$ and let w belong to D^- . Then, the spaces $\mathcal{L}_w(\beta)$ and $\mathcal{L}_w^-(\beta)$ are equal, and their norms are equivalent.*

The next theorem gives us a characterization of the spaces $\mathcal{L}_w(\beta)$, $0 \leq \beta < 1$, whenever w belongs to A_p^- . In the case $\beta = 0$, we shall prove this result using Proposition 3.6, which states a one-sided weighted version of John-Nirenberg Inequality.

Theorem 2.2. *Let $0 \leq \beta < 1$ and $1 \leq p < \infty$. Let w be a weight such that w belongs to A_p^- . Then, $f \in \mathcal{L}_w(\beta)$ if and only if there exists a constant C such that*

$$(2.1) \quad \int_{I^-} |f(x) - f_{I^+}|^q w(x)^{1-q} dx \leq C w(I^-) |I|^{\beta q},$$

for all bounded interval I and every $q : 1 \leq q \leq p', q < \infty$.

The following two theorems state a sufficient and necessary condition on the weight w to obtain extensions of I_α^+ defined on certain spaces.

Theorem 2.3. *Let $0 < \alpha < 1$, $1 < p < \infty$ and $\beta = \alpha - 1/p$. The following statements are equivalent.*

- (i) *The weight w belongs to $H^-(\alpha, p)$.*
- (ii) *The operator I_α^+ can be extended to a linear bounded operator $\widetilde{I}_\alpha^+$ from $\widetilde{L}_w^p$ into $\mathcal{L}_w^-(\beta)$ by means of*

$$(2.2) \quad \begin{aligned} & \widetilde{I}_\alpha^+(f)(x) \\ &= - \int_{x_0}^x \frac{f(y) dy}{|y - x|^{1-\alpha}} + \int_{x_0}^\infty \left[\frac{1}{|y - x|^{1-\alpha}} - \frac{1 - \chi_{[x_0, x_0+1]}(y)}{(y - x_0)^{1-\alpha}} \right] f(y) dy, \end{aligned}$$

for an appropriate choice of $x_0 \in \mathbb{R}$.

- (iii) *The operator I_α^+ can be extended to a linear bounded operator $\widetilde{I}_\alpha^+$ from L_w^p into $\mathcal{L}_w^-(\beta)$, where $\widetilde{I}_\alpha^+$ is defined as in (2.2).*

Theorem 2.4. *Let w a weight and $0 < \alpha < 1$. The following statements are equivalent.*

- (i) *$w \in H^-(\alpha, \infty)$.*

(ii) The operator I_α^+ can be extended to a linear bounded operator $\widetilde{I}_\alpha^+ : \mathcal{L}_w(0) \longrightarrow \mathcal{L}_w(\alpha)$ by means of

$$\widetilde{I}_\alpha^+(f)(x) = \int_{-\infty}^{\infty} \left[\frac{\chi_{[x_0, \infty)}(y)}{|y - x_0|^{1-\alpha}} - \frac{\chi_{[x, \infty)}(y)}{|y - x|^{1-\alpha}} \right] f(y) dy,$$

for an appropriate choice of $x_0 \in \mathbb{R}$.

Remark 2.5. Let $1 < p < \frac{1}{\alpha}$ and $\beta = \alpha - 1/p < 0$.

(i) It is easy to see that if w belongs to $RH^-(\frac{1}{1+\beta})$, then $L_w^{-1/\beta} \subseteq \mathcal{L}_w^-(\beta)$.

(ii) By Lemma 4.4 in [9], if $w^{p'}$ belongs to $A_{-\beta p'+1}^-$ then w satisfies the condition $RH^-(p')$, and taking into account that $\frac{1}{1+\beta} < p'$, it follows that w belongs to $RH^-(\frac{1}{1+\beta})$.

(iii) Theorem 6 in [4] states the fact that $w^{p'}$ belongs to $A_{-\beta p'+1}^-$ is a necessary and sufficient condition for the boundedness of I_α^+ from L_w^p into $L_w^{-1/\beta} \subseteq \mathcal{L}_w^-(\beta)$.

(iv) If $w^{p'}$ belongs to $A_{-\beta p'+1}^-$, since $w^{p'} \in A_{p'+1}^-$, we have that w belongs to $H^-(\alpha, p)$. However, there exist weights w belonging to $H^-(\alpha, p)$ such that $w^{p'}$ does not belong to $A_{p'+1}^-$, for example, $w(x) = |x|^\gamma$ for $-\beta \leq \gamma < 1 - \beta$, see Remark 4.3.

In consequence, if $-1 < \beta < 0$ and $w^{p'}$ belongs to $A_{-\beta p'+1}^-$, the extension of I_α^+ in Theorem 2.3 can be obtained from Theorem 6 in [4]. But, (iv) shows that Theorem 2.3 can be applied to a larger class of weights.

Remark 2.6. Let w be a weight. We shall say that a locally integrable function f defined on $\mathbb{R}$, belongs to $MW^-(w)$ if there exists a constant C such that

$$\frac{1}{|I|} \frac{1}{\text{ess inf}_{I-w}} \int_I |f(y) - f_I| dy \leq C,$$

for every bounded interval I .

(i) By Definition 1.2, it follows that $MW^-(w) \subseteq \mathcal{L}_w^-(0)$. Moreover, if w belongs to A_1^- then $\mathcal{L}_w(0) \subseteq MW^-(w)$, and as a consequence of Theorem 2.1, $\mathcal{L}_w^-(0) = MW^-(w)$.

(ii) Following the same lines of Theorem 7 in [7], it can be seen that, in the case $\alpha = 1/p$, the weight $w^{p'}$ belongs to A_1^- if and only if the operator I_α^+ is bounded from L_w^p into $MW^-(w)$. Also see [2].

(iii) If $w^{p'}$ belongs to A_1^- then, by Remark 4.3, w belongs to $H^-(\alpha, p)$.

In consequence, the fact that $w^{p'}$ belongs to A_1^- implies the boundedness of I_α^+ from L_w^p into $MW^-(w)$, is contained in Theorem 2.3.

3. THE SPACES $\mathcal{L}_w(\beta)$ AND $\mathcal{L}_w^-(\beta)$

The next lemma will be used in the proof of Theorem 2.1.

Lemma 3.1. *Let $-1 < \beta < 1$, f a locally integrable function defined on $\mathbb{R}$, and $w \in D^-$. The following statements are equivalent.*

- (i) $f \in \mathcal{L}_w^-(\beta)$.
- (ii) *There exists a constant C such that for every $a \in \mathbb{R}$ and $h > 0$,*

$$\frac{1}{w([a - h/2, a])h^\beta} \int_a^{a+h} |f(y) - f_{[a+h/2, a+h]}| dy \leq C.$$

- (iii) *There exists a constant C such that for every $a \in \mathbb{R}$ and $h > 0$,*

$$\frac{1}{w([a - h/2, a])h^\beta} \int_a^{a+h} |f(y) - f_{[a+h, a+3h]}| dy \leq C.$$

The constants C in (ii) and (iii) are equivalent to $\|f\|_{\mathcal{L}_w^-(\beta)}$.

Proof.

(i) $\Rightarrow$ (ii). Using (i) and taking into account that $w \in D^-$, we have

$$\begin{aligned} & \int_a^{a+h/2} |f(y) - f_{[a+h/2, a+h]}| dy \\ & \leq \int_a^{a+h/2} |f(y) - f_{[a+h/4, a+h/2]}| dy + 2 \int_{a+h/4}^{a+h} |f(y) - f_{[a+h/2, a+h]}| dy \\ & \leq 3 \int_a^{a+h/2} |f(y) - f_{[a, a+h/2]}| dy + 5 \int_{a+h/4}^{a+h} |f(y) - f_{[a+h/4, a+h]}| dy \\ & \leq C \|f\|_{\mathcal{L}_w^-(\beta)} w([a - h/2, a]) h^\beta + C \|f\|_{\mathcal{L}_w^-(\beta)} w([a - h/2, a + h/4]) h^\beta \\ & \leq C \|f\|_{\mathcal{L}_w^-(\beta)} w([a - h/2, a]) h^\beta. \end{aligned}$$

From these inequalities and using (i) again, we have the estimate

$$\begin{aligned} & \int_a^{a+h} |f(y) - f_{[a+h/2, a+h]}| dy \\ & = \int_a^{a+h/2} |f(y) - f_{[a+h/2, a+h]}| dy + \int_{a+h/2}^{a+h} |f(y) - f_{[a+h/2, a+h]}| dy \\ & \leq C \|f\|_{\mathcal{L}_w^-(\beta)} w([a - h/2, a]) h^\beta + C \|f\|_{\mathcal{L}_w^-(\beta)} w([a, a + h/2]) h^\beta \\ & \leq C \|f\|_{\mathcal{L}_w^-(\beta)} w([a - h/2, a]) h^\beta, \end{aligned}$$

which shows that (ii) holds. In a similar way it can be proved that (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i). $\square$

As we have already mentioned if w belongs to D^- then, for every $-1 < \beta < 1$ we have the inclusion $\mathcal{L}_w(\beta) \subseteq \mathcal{L}_w^-(\beta)$. In order to prove Theorem 2.1, it will be sufficient to show that $\mathcal{L}_w^-(\beta) \subseteq \mathcal{L}_w(\beta)$.

Proof of Theorem 2.1. We suppose that $f \in \mathcal{L}_w^-(\beta)$. Let $a \in \mathbb{R}$ and $h > 0$. For each $j \geq 0$ we define $a_j = a + h/2^j$. Then,

$$\begin{aligned}
 (3.1) \quad & \int_a^{a+h/2} |f(y) - f_{[a+h/2, a+h]}| dy = \sum_{j=1}^{\infty} \int_{a_{j+1}}^{a_j} |f(y) - f_{[a+h/2, a+h]}| dy \\
 & \leq \sum_{j=1}^{\infty} \int_{a_{j+1}}^{a_j} |f(y) - f_{[a_j, a_{j-1}]}| dy + \sum_{j=2}^{\infty} \frac{h}{2^{j+1}} |f_{[a_j, a_{j-1}]} - f_{[a_1, a_0]}| \\
 & = I + II.
 \end{aligned}$$

Taking into account that for each $j \geq 2$,

$$|f_{[a_j, a_{j-1}]} - f_{[a_1, a_0]}| \leq \frac{2^j}{h} \int_{a_j}^{a_{j-1}} |f - f_{[a+h/2, a+h]}|$$

it follows that,

$$II \leq \sum_{j=2}^{\infty} \frac{1}{2} \int_{a_j}^{a_{j-1}} |f - f_{[a+h/2, a+h]}| = \frac{1}{2} \int_a^{a+h/2} |f(y) - f_{[a+h/2, a+h]}| dy.$$

Then, by (3.1)

$$(3.2) \quad \int_a^{a+h/2} |f(y) - f_{[a+h/2, a+h]}| dy \leq 2I.$$

Now, using part (iii) of Lemma 3.1 and keeping in mind that $\beta \geq 0$ we have that,

$$(3.3) \quad I \leq C \sum_{j=1}^{\infty} \left(\frac{h}{2^j} \right)^{\beta} w([a_{j+2}, a_{j+1}]) \leq Ch^{\beta} w([a, a + h/4]).$$

From (3.2) and (3.3), and taking into account that $f \in \mathcal{L}_w^-(\beta)$, we get

$$\begin{aligned}
 & \int_a^{a+h} |f(y) - f_{[a+h/2, a+h]}| dy \\
 & = \int_a^{a+h/2} |f(y) - f_{[a+h/2, a+h]}| dy + \int_{a+h/2}^{a+h} |f(y) - f_{[a+h/2, a+h]}| dy \\
 & \leq Ch^{\beta} w([a, a + h/4]) + Ch^{\beta} w([a, a + h/2]) \\
 & \leq Ch^{\beta} w([a, a + h]).
 \end{aligned}$$

Therefore,

$$\int_a^{a+h} |f(y) - f_{[a, a+h]}| dy \leq 3 \int_a^{a+h} |f(y) - f_{[a+h/2, a+h]}| dy \leq Ch^{\beta} w([a, a + h]),$$

which shows that $f \in \mathcal{L}_w(\beta)$. $\square$

Remark 3.2. Let $-1 < \beta < 0$ and $w(t) = e^{-t}$. The weight w belongs to A_1^- however, we only have the strict inclusion $\mathcal{L}_w(\beta) \subset \mathcal{L}_w^-(\beta)$. For example, given $a > 1$ we consider the function

$$f(t) = \begin{cases} e^{-at} & , \quad t \geq 0 \\ 1 & , \quad t < 0. \end{cases}$$

We observe, using Remark 2.5 part (i), that $f \in \mathcal{L}_w^-(\beta)$. On the other hand,

$$\begin{aligned} \frac{1}{h^\beta w([0, h])} \int_0^h |f - f_{[h, 2h]}| &= \frac{1}{h^\beta (1 - e^{-h})} \left[\frac{1 - e^{-ah}}{a} - \frac{e^{-ah}}{a} (1 - e^{-ah}) \right] \\ &= \frac{(1 - e^{-ah})^2}{h^\beta (1 - e^{-h}) a}, \end{aligned}$$

which tends to infinite whenever h tends to infinite. This implies that $f \notin \mathcal{L}_w(\beta)$.

The next proposition will be used in the proof of Theorem 2.2.

Proposition 3.3. Let $0 < \beta < 1$ and let w belong to D^- . Then, $f \in \mathcal{L}_w(\beta)$ if and only if, there exists a constant C such that

(3.4)

$$|f(x) - f(y)| \leq C \left[\int_x^{x + \frac{|y-x|}{2}} \frac{w(z)}{(z-x)^{1-\beta}} dz + \int_y^{y + \frac{|y-x|}{2}} \frac{w(z)}{(z-y)^{1-\beta}} dz \right],$$

for almost every real numbers x and y .

Proof. We suppose that $f \in \mathcal{L}_w(\beta)$. We shall show that for every $h > 0$ and almost every x ,

$$(3.5) \quad |f(x) - f_{[x+h/2, x+h]}| \leq C \int_x^{x+h/2} \frac{w(z)}{(z-x)^{1-\beta}} dz.$$

For each $i \geq 0$ let $x_i = x + h/2^i$. If x is a Lebesgue point of f we have that,

$$\begin{aligned} (3.6) \quad &|f(x) - f_{[x+h/2, x+h]}| \\ &\leq |f(x) - f_{[x_{i+1}, x_i]}| + |f_{[x_{i+1}, x_i]} - f_{[x_1, x_0]}| \\ &\leq |f(x) - f_{[x_{i+1}, x_i]}| + \sum_{j=1}^i |f_{[x_{j+1}, x_j]} - f_{[x_j, x_{j-1}]}| \\ &\leq \sum_{j=1}^{\infty} |f_{[x_{j+1}, x_j]} - f_{[x_j, x_{j-1}]}|. \end{aligned}$$

For each $j \geq 1$, since $f \in \mathcal{L}_w(\beta)$ we obtain

$$|f_{[x_{j+1}, x_j]} - f_{[x_j, x_{j-1}]}| \leq C \frac{1}{(x_{j+1} - x_{j-1})^{1-\beta}} w([x_{j+1}, x_{j-1}]).$$

From this inequality, (3.6) and taking into account that $w \in D^-$ we get,

$$\begin{aligned} |f(x) - f_{[x+h/2, x+h]}| &\leq C \sum_{j=1}^{\infty} \int_{x_{j+1}}^{x_{j-1}} \frac{w(z)}{(z-x)^{1-\beta}} dz \\ &= C \int_x^{x+h} \frac{w(z)}{(z-x)^{1-\beta}} dz \leq C \int_x^{x+h/2} \frac{w(z)}{(z-x)^{1-\beta}} dz, \end{aligned}$$

which shows that (3.5) holds. Let $x < y$ two Lebesgue points of f . By (3.5) we have that,

$$\begin{aligned} (3.7) \quad |f(x) - f(y)| &\leq |f(x) - f_{[\frac{x+y}{2}, y]}| + |f(y) - f_{[y+\frac{y-x}{2}, y+(y-x)]}| \\ &\quad + |f_{[\frac{x+y}{2}, y]} - f_{[y+\frac{y-x}{2}, y+(y-x)]}| \\ &\leq C \left[\int_x^{x+\frac{|y-x|}{2}} \frac{w(z)}{(z-x)^{1-\beta}} dz + \int_y^{y+\frac{|y-x|}{2}} \frac{w(z)}{(z-y)^{1-\beta}} dz \right] \\ &\quad + |f_{[\frac{x+y}{2}, y]} - f_{[y+\frac{y-x}{2}, y+(y-x)]}|. \end{aligned}$$

From the hypotheses $f \in \mathcal{L}_w(\beta)$ and $w \in D^-$, it follows that the third term on the right hand is bounded by

$$\begin{aligned} &\frac{C}{y-x} \int_{x+\frac{y-x}{2}}^{y+(y-x)} |f(t) - f_{[x+\frac{y-x}{2}, y+(y-x)]}| dt \\ &\leq \frac{C}{(y-x)^{1-\beta}} w([x, y + \frac{y-x}{2}]) \leq C \int_x^{(x+y)/2} \frac{w(z)}{(z-x)^{1-\beta}} dz. \end{aligned}$$

Therefore, by (3.7) we have that (3.4) holds.

Conversely, given a real number a and $h > 0$, by (3.4)

$$\begin{aligned} (3.8) \quad &\int_a^{a+h} |f(x) - f_{[a, a+h]}| dx \\ &\leq C \left[\int_a^{a+h} \int_x^{x+\frac{|y-x|}{2}} \frac{w(z)}{(z-x)^{1-\beta}} dz dx + \int_a^{a+h} \int_y^{y+\frac{|y-x|}{2}} \frac{w(z)}{(z-y)^{1-\beta}} dz dy \right]. \end{aligned}$$

Changing the order of integration and taking into account that $w \in D^-$, it follows that (3.8) is bounded by $Ch^\beta w([a, a+h])$. This completes the proof of the proposition. $\square$

The next two lemmas will be needed in the proof of Propotion 3.6.

Lemma 3.4. *Let $w \in D^-$ and $f \in \mathcal{L}_w(0)$. Given two intervals $I \subseteq J$ the inequality*

$$\frac{1}{w(J)} \int_J |f - f_{I^+}| \chi_{I^- \cup I} \leq C \|f\|_{\mathcal{L}_w(0)},$$

holds with a constant C only depending on w .

Proof. Let $I = (a, b)$ and $J = (c, d)$. We consider $\alpha = \max\{a - |I|, c\}$ and $\beta = b + |I|$. Since $J \cap (I^- \cup I) \subseteq (\alpha, \beta)$ we have that,

$$\begin{aligned} (3.9) \quad \frac{1}{w(J)} \int_J |f - f_{I^+}| \chi_{I^- \cup I} &\leq \frac{1}{w(J)} \int_{\alpha}^{\beta} |f - f_{I^+}| \\ &\leq \frac{1}{w(J)} \left[\int_{\alpha}^{\beta} |f - f_{(\alpha, \beta)}| + \frac{(\beta - \alpha)}{|I^+|} \int_{I^+} |f - f_{(\alpha, \beta)}| \right] \end{aligned}$$

We observe that $(\beta - \alpha) \leq 3|I|$, which implies

$$(3.9) \leq \frac{4}{w(J)} \int_{\alpha}^{\beta} |f - f_{(\alpha, \beta)}|.$$

From the hypotheses $f \in \mathcal{L}_w(0)$ and $w \in D^-$, and taking into account that $(\alpha, \beta) \subseteq J \cup J^+$, (3.9) is bounded by

$$\frac{4}{w(J)} \|f\|_{\mathcal{L}_w(0)} w((\alpha, \beta)) \leq C \|f\|_{\mathcal{L}_w(0)},$$

as we wanted to prove. $\square$

Lemma 3.5. *Let $1 < p < \infty$ and $w \in A_p^-$. Then, there exists a constant C such that for every $\beta > 0$ the inequality*

$$(3.10) \quad w(\{x \in I^- : w(x) < \beta\}) \leq C \left[\beta \frac{|I^+|}{w(I^+)} \right]^{p'} w(I^+),$$

holds.

Proof. This lemma is a simple variant of Lemma 3.1 in [6]. $\square$

The following proposition is a one-sided weighted version of John-Nirenberg Inequality. For its proof we shall use the method employed in Theorem 3 in [6] and the techniques of Lemma 1 in [5].

Proposition 3.6. *Let f belong to $\mathcal{L}_w(0)$. Then,*

(i) if $w \in A_1^-$ there exist positive constants C_1 y C_2 such that for every $\lambda > 0$,

$$w(\{x \in I^- : |f(x) - f_{I^+}| w(x)^{-1} > \lambda\}) \leq C_1 e^{-C_2 \lambda / \|f\|_{\mathcal{L}_w(0)}} w(I^-)$$

holds for every bounded interval I .

(ii) if $w \in A_p^-, 1 < p < \infty$ there exists a positive constant C_3 such that for every $\lambda > 0$,

$$w(\{x \in I^- : |f(x) - f_{I^+}|w(x)^{-1} > \lambda\}) \leq C_3 (1 + \lambda/\|f\|_{\mathcal{L}_w(0)})^{-p'} w(I^-)$$

holds for every bounded interval I .

Proof. Without loss of generality we can suppose that $\|f\|_{\mathcal{L}_w(0)} = 1$. For each $\lambda > 0$ and each bounded interval I , let

$$A(\lambda, I) = w(\{x \in I^- : |f(x) - f_{I^+}|w(x)^{-1} > \lambda\}),$$

and

$$(3.11) \quad \mathcal{A}(\lambda) = \sup \frac{A(\lambda, I)}{w(I^-)},$$

where the supremum is taken over all $f : \|f\|_{\mathcal{L}_w(0)} = 1$, and all bounded interval I . Thus, for every $\lambda > 0$, we have that $\mathcal{A}(\lambda) \leq 1$.

By Lemma 3.4 there exists a constant μ satisfying

$$(3.12) \quad \frac{1}{w(J)} \int_J |f - f_{I^+}| \chi_{I^- \cup I} \leq \mu,$$

for every bounded intervals $I \subseteq J$ and every $f : \|f\|_{\mathcal{L}_w(0)} = 1$.

Fixed $I = [a, b]$, let $s > \mu$ and

$$\Omega_s = \{x : M_w^-(|f - f_{I^+}| \chi_{I^- \cup I} w^{-1})(x) > s\}.$$

Since Ω_s is an open set, we can write $\Omega_s = \cup_{i \geq 1} J_i$, where the J_i 's are its connected components.

We observe that if $J_i \cap I^- \neq \emptyset$ then $J_i \cap I^+ = \emptyset$. In fact, suppose that $J_i \cap I^- \neq \emptyset$ and let $J_i = (\alpha, \beta)$. If $\beta \geq b$ a simple variant of Lemma 2.1 in [12], shows that

$$\mu < s \leq \frac{1}{w((\alpha, b))} \int_{\alpha}^b |f - f_{I^+}| \chi_{I^- \cup I}.$$

However, using (3.12) we have that

$$\frac{1}{w((\alpha, b))} \int_{\alpha}^b |f - f_{I^+}| \chi_{I^- \cup I} \leq \mu.$$

In consequence, $\beta < b$ and $J_i \cap I^+ = \emptyset$.

Let $\{J_i : J_i \cap I^- \neq \emptyset\} = \{H_i\}_{i \geq 1}$. For each i ,

$$(3.13) \quad H_i \subseteq I^- \cup I \quad \text{and} \quad \frac{1}{w(H_i)} \int_{H_i} |f - f_{I^+}| = s.$$

By Lebesgue's Differentiation Theorem with respect to w for almost every $x \in I^- \setminus \cup_{i \geq 1} H_i$,

$$|f(x) - f_{I^+}|w(x)^{-1} \leq s.$$

Using (3.13), (3.12) and keeping in mind that $w \in D^-$, we have that

$$(3.14) \quad \sum_{i \geq 1} w(H_i) = \frac{1}{s} \sum_{i \geq 1} \int_{H_i} |f - f_{I+}| \leq \frac{1}{s} \int_{I^- \cup I} |f - f_{I+}|$$

$$\leq \frac{1}{s} \mu w(I^- \cup I) \leq \frac{1}{s} \mu C_w w(I^-).$$

Fixed $H_i = (a_i, b_i)$ we define the sequences $(x_k)_{k \geq 1}$ and $(y_k)_{k \geq 1}$ by $b_i - x_k = 2(b_i - y_k) = (2/3)^k |H_i|$, and the intervals $H_{i,k} = (x_k, y_k)$. Therefore,

$$(3.15) \quad H_i = \bigcup_{k \geq 1} H_{i,k}^- \quad , \quad \frac{1}{w(H_{i,k}^+)} \int_{H_{i,k}^+} |f - f_{I+}| \leq s,$$

and

$$|f(x) - f_{I+}| w(x)^{-1} \leq \lambda \quad a.e. \ x \in I^- \setminus \bigcup_{k,i} H_{i,k}^-.$$

Then,

$$A(\lambda, I) \leq \sum_{i,k} w(\{x \in H_{i,k}^- : |f(x) - f_{I+}| w(x)^{-1} > \lambda\}).$$

If $\mu < s \leq \lambda$ and $0 < \gamma < \lambda$, we have that

$$(3.16) \quad A(\lambda, I) \leq \sum_{i,k} w(\{x \in H_{i,k}^- : |f(x) - f_{H_{i,k}^+}| w(x)^{-1} > \lambda - \gamma\})$$

$$+ \sum_{i,k} w(\{x \in H_{i,k}^- : |f_{H_{i,k}^+} - f_{I+}| w(x)^{-1} > \gamma\})$$

$$= I + II.$$

From (3.11), (3.15) and (3.14) we obtain the estimate

$$(3.17) \quad I \leq \sum_{i,k} \mathcal{A}(\lambda - \gamma) w(H_{i,k}^-) = \mathcal{A}(\lambda - \gamma) \sum_i w(H_i)$$

$$\leq \frac{C_w \mu}{s} \mathcal{A}(\lambda - \gamma) w(I^-).$$

On the other hand, (3.15) implies that

$$(3.18) \quad |f_{H_{i,k}^+} - f_{I+}| \leq \frac{1}{|H_{i,k}^+|} \int_{H_{i,k}^+} |f - f_{I+}| \leq s \frac{w(H_{i,k}^+)}{|H_{i,k}^+|}.$$

If $w \in A_1^-$ there exists $\rho > 1$ such that for every i, k and almost every $x \in H_{i,k}^-$,

$$\frac{w(H_{i,k}^+)}{|H_{i,k}^+|} \leq \rho w(x).$$

Then, using (3.18) we have

$$|f_{H_{i,k}^+} - f_{I^+}| \leq \rho \operatorname{ess\,inf}_{x \in H_{i,k}^-} w(x).$$

In consequence,

$$\begin{aligned} & w(\{x \in H_{i,k}^- : |f_{H_{i,k}^+} - f_{I^+}| w(x)^{-1} > \gamma\}) \\ & \leq w(\{x \in H_{i,k}^- : w(x) < \frac{\rho s}{\gamma} \operatorname{ess\,inf}_{x \in H_{i,k}^-} w(x)\}). \end{aligned}$$

Choosing $s = 2\mu C_w$ y $\gamma = \rho s$, if $\lambda > \gamma$ we have $\mu < s < \lambda$ and $II = 0$. Then, from (3.16) and (3.17) we obtain that

$$A(\lambda, I) \leq \frac{1}{2} \mathcal{A}(\lambda - \gamma) w(I^-),$$

that is, if $\lambda > \gamma$,

$$\mathcal{A}(\lambda) \leq \frac{1}{2} \mathcal{A}(\lambda - \gamma).$$

Now, proceeding as in Theorem 3 of [6], it can be obtained part (i) of this proposition.

In order to prove part (ii), we suppose that $w \in A_p^-, 1 < p < \infty$. Using (3.18), Lemma 3.5 and taking into account that $w \in D^-$

$$\begin{aligned} & w(\{x \in H_{i,k}^- : |f_{H_{i,k}^+} - f_{I^+}| w(x)^{-1} > \gamma\}) \\ & \leq w\left(\left\{x \in H_{i,k}^- : w(x) < \frac{s w(H_{i,k}^+)}{\gamma |H_{i,k}^+|}\right\}\right) \leq C \left[\frac{s w(H_{i,k}^+)}{\gamma |H_{i,k}^+|} \frac{|H_{i,k}|}{w(H_{i,k})}\right]^{p'} w(H_{i,k}) \\ & \leq C \left(\frac{s}{\gamma}\right)^{p'} w(H_{i,k}^-). \end{aligned}$$

By (3.15) and (3.14), we have

$$II \leq C \left(\frac{s}{\gamma}\right)^{p'} \sum_{i,k} w(H_{i,k}^-) = C \left(\frac{s}{\gamma}\right)^{p'} \sum_i w(H_i) \leq C \mu \frac{s^{p'-1}}{\gamma^{p'}} w(I^-).$$

Then, (3.16) and (3.17) imply that

$$A(\lambda, I) \leq C \mu \left[\frac{A(\lambda - \gamma)}{s} + \frac{s^{p'-1}}{\gamma^{p'}} \right] w(I^-).$$

From this inequality, part (ii) follows as in Theorem 3 of [6]. $\square$

Proposition 3.7. *Let $0 < \beta < 1$ and $1 < p < \infty$. Let w be a weight such that $w^{1+\frac{\beta}{1-\beta p}}$ belongs to A_p^- . Then, $f \in \mathcal{L}_w(\beta)$ if and only if there exists a constant C such that (2.1) holds for all bounded interval I and every $q : 1 \leq q \leq p'/(1-\beta)$.*

Proof. Suppose that (2.1) holds for every $q : 1 \leq q \leq p'/(1-\beta)$. Taking $q = 1$ it is easy to show that $f \in \mathcal{L}_w(\beta)$. Conversely, let f belong to $\mathcal{L}_w(\beta)$. We observe that it will be sufficient to consider $q = p'/(1-\beta)$, because from this case and applying Hölder's inequality we obtain (2.1) for every $1 \leq q < p'/(1-\beta)$. Given a bounded interval I and using Proposition 3.3, we have that

$$\begin{aligned}
 (3.19) \quad & \int_{I^-} |f(x) - f_{I^+}|^q w(x)^{1-q} dx \leq \int_{I^-} \left[\frac{1}{|I^+|} \int_{I^+} |f(x) - f(y)| dy \right]^q w(x)^{1-q} dx \\
 & \leq C \int_{I^-} w(x)^{1-q} \left[\frac{1}{|I^+|} \int_{I^+} \left(\int_x^{x+\frac{|y-x|}{2}} \frac{w(z)}{(z-x)^{1-\beta}} dz + \int_y^{y+\frac{|y-x|}{2}} \frac{w(z)}{(z-y)^{1-\beta}} dz \right) dy \right]^q dx \\
 & \leq C \int_{I^-} w(x)^{1-q} \left(\int_x^{x+\frac{3|I|}{2}} \frac{w(z)}{(z-x)^{1-\beta}} dz \right)^q dx \\
 & \quad + \frac{C}{|I^+|^q} \int_{I^-} w(x)^{1-q} \left(\int_{I^+} \int_y^{y+\frac{3|I|}{2}} \frac{w(z)}{(z-y)^{1-\beta}} dz dy \right)^q dx \\
 & = A + B.
 \end{aligned}$$

If we denote $J = I^- \cup I \cup I^+$ then we have the estimate

$$A \leq C \int_{I^-} w(x)^{1-q} I_\beta^+(w\chi_J)(x)^q dx.$$

Our hypothesis $w^{1+\frac{\beta}{1-\beta}p} \in A_p^-$ is equivalent to

$$(3.20) \quad w^{1-\frac{p'}{1-\beta}} \in A_{p'}^+,$$

where $p' = 1 + \frac{q}{s'}$ and $\frac{1}{s} = \frac{1}{q} + \beta$. Then, by Theorem 6 in [4] it follows that

$$A \leq C \left(\int_{-\infty}^{\infty} w(x)^{-\frac{s}{q'}} |w\chi_J(x)|^s dx \right)^{q/s} = C \left(\int_J w(x)^{s/q} dx \right)^{q/s}.$$

Since $q/s = q\beta + 1 > 1$, applying Hölder's inequality and taking into account that $w \in D^-$ we obtain

$$(3.21) \quad A \leq C \int_J w(x) dx |J|^{\frac{q}{s}-1} \leq C w(I^-) |I|^{\beta q}.$$

Let us estimate B . If we set $J' = I^+ \cup I^{++} \cup I^{+++}$, then

$$B \leq \frac{C}{|I^+|^q} \int_{I^-} w(x)^{1-q} \left(\int_{I^+} I_\beta^+(w\chi_{J'})(y) dy \right)^q dx$$

Applying Hölder's inequality,

$$B \leq \frac{C}{|I^+|^q} \left(\int_{I^-} w(x)^{1-q} dx \right) \left(\int_{I^+} w(y) dy \right)^{q/q'} \int_{I^+} w(y)^{1-q} I_\beta^+(w\chi_{J'})(y)^q dx.$$

From (3.20), it follows that $w^{1-q} \in A_q^+$ then, we have that

$$B \leq C \int_{I^+} w(y)^{1-q} I_\beta^+(w\chi_{J'})(y)^q dx.$$

Proceeding as in the estimation of A and taking into account that $w \in D^-$ we obtain

$$(3.22) \quad B \leq Cw(I^-)|I|^{\beta q}.$$

As consequence of (3.19), (3.21) and (3.22) we get (2.1) and the proof of this proposition is complete. $\square$

Proof of Theorem 2.2. We shall prove that f belonging to $\mathcal{L}_w(\beta)$ is a sufficient condition for (2.1) holds. The fact that (2.1) is a necessary condition follows as in the previous proposition. For that, we shall consider different cases.

First of all, we assume that $\beta = 0$ and $f \in \mathcal{L}_w(0)$. If $w \in A_1^-$ we have that (2.1) is an immediate consequence of Proposition 3.6 part (i). If $w \in A_p^-$, $1 < p < \infty$, we have that $w \in A_{p-\epsilon}^-$ for some $\epsilon > 0$. Then, by Proposition 3.6 part (ii), and proceeding as in Theorem 4 of [6], we obtain that f satisfies (2.1).

Let $0 < \beta < 1$ and $1 < p < \infty$. Since the weight w belongs to A_p^- there exists $0 < \alpha < \beta$ such that $w^{1+\frac{\alpha}{1-\alpha}p}$ belongs to A_p^- . Proceeding as in (3.19), we have that

$$\begin{aligned} & \int_{I^-} |f(x) - f_{I^+}|^q w(x)^{1-q} dx \\ & \leq C \int_{I^-} w(x)^{1-q} \left[\frac{1}{|I^+|} \int_{I^+} \left(\int_x^{x+\frac{|y-x|}{2}} \frac{w(z)}{(z-x)^{1-\beta}} dz + \int_y^{y+\frac{|y-x|}{2}} \frac{w(z)}{(z-y)^{1-\beta}} dz \right) dy \right]^q dx \\ & \leq C|I|^{(\beta-\alpha)q} \int_{I^-} w(x)^{1-q} \left(\int_x^{x+\frac{3|I|}{2}} \frac{w(z)}{(z-x)^{1-\alpha}} dz \right)^q dx \\ & \quad + \frac{C}{|I|^{(\beta-\alpha-1)q}} \int_{I^-} w(x)^{1-q} \left(\int_{I^+} \int_y^{y+\frac{3|I|}{2}} \frac{w(z)}{(z-y)^{1-\alpha}} dz dy \right)^q dx \\ & = |I|^{(\beta-\alpha)q} (A + B). \end{aligned}$$

Substituting in the proof of the previous proposition α for β in the estimation of A and B we obtain this case.

Finally, we suppose that $0 < \beta < 1$ and $p = 1$. Since the weight w belongs to A_1^- it follows that w belongs to A_s^- for every $1 < s < \infty$. Then, by the previous case we obtain that (2.1) holds for every $1 \leq q < \infty$. $\square$

4. THE CLASSES $H^-(\alpha, p)$

The next lemma states necessary conditions for that a weight w belongs to $H^-(\alpha, p)$.

Lemma 4.1. *Let $1 < p \leq \infty$. If $w \in H^-(\alpha, p)$ then,*

- (i) $w^{p'}$ belongs to D^- ,
- (ii) w belongs to $\in RH^-(p')$,
- (iii) w belongs to D^- .

Proof. The proof of parts (i) and (ii) are similar to ones of Lemma 3.7 and Lemma 3.8, in [1], respectively. Applying Hölder's inequality and (ii), we obtain (iii). $\square$

Lemma 4.2. *Let w be a weight. The following conditions are equivalent.*

- (a) $w \in H^-(\alpha, p)$.
- (b) $w \in RH^-(p')$ and there exist positive constants C and ϵ such that,

$$w^{p'}([a, a + \theta t]) \leq C \theta^{(2-\alpha)p' - \epsilon} w^{p'}([a, a + t]),$$

for every $a \in \mathbb{R}$, $t > 0$ and $\theta \geq 1$.

- (c) There exist positive constants C and ϵ such that,

$$\left(\frac{w^{p'}([a, a + \theta t])}{\theta t} \right)^{1/p'} \leq C \theta^{\frac{1}{p} + 1 - \alpha - \frac{\epsilon}{p'}} \frac{w([a - t, a])}{t},$$

for every $a \in \mathbb{R}$, $t > 0$ and $\theta \geq 1$.

Proof.

(a) $\Rightarrow$ (b) : By Lemma 4.1 (ii) we have that $w \in RH^-(p')$.

Let $I = [a, a + t]$. Applying Hölder's inequality and keeping in mind that $w \in H^-(\alpha, p)$,

$$\begin{aligned} (4.1) \quad \frac{w^{p'}(I)}{|I|} &\geq \left(\frac{w(I)}{|I|} \right)^{p'} \geq C |I|^{(\frac{1}{p} - \alpha + 1)p'} \int_{a+t}^{\infty} \frac{w(y)^{p'}}{(y-a)^{(2-\alpha)p'}} dy \\ &\geq C |I|^{(\frac{1}{p} - \alpha + 1)p'} \sum_{k \geq 0} \frac{1}{(2^{k+1}t)^{(2-\alpha)p'}} \int_{a+2^k t}^{a+2^{k+1}t} w(y)^{p'} dy. \end{aligned}$$

Since $\sum_{i \geq k} \left(\frac{1}{2^{(2-\alpha)p'}} \right)^i = C \left(\frac{1}{2^{(2-\alpha)p'}} \right)^k$, by (4.1) and applying Fubini's Theorem,

$$\begin{aligned} \frac{w^{p'}(I)}{|I|} &\geq C|I|^{(\frac{1}{p}-\alpha+1)p'} \frac{1}{t^{(2-\alpha)p'}} \sum_{k \geq 0} \int_{a+2^k t}^{a+2^{k+1}t} w(y)^{p'} dy \sum_{i \geq k} \left(\frac{1}{2^{(2-\alpha)p'}} \right)^i \\ &= C|I|^{(\frac{1}{p}-\alpha+1)p'} \sum_{i \geq 0} \frac{1}{(2^i t)^{(2-\alpha)p'}} \sum_{k=0}^i \int_{a+2^k t}^{a+2^{k+1}t} w(y)^{p'} dy \\ &= C|I|^{(\frac{1}{p}-\alpha+1)p'} \sum_{i \geq 0} \frac{1}{(2^i t)^{(2-\alpha)p'}} \int_{a+t}^{a+2^{i+1}t} w(y)^{p'} dy. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{w^{p'}(I)}{|I|} &\geq C|I|^{(\frac{1}{p}-\alpha+1)p'} \sum_{i \geq 0} \frac{1}{(2^i t)^{(2-\alpha)p'}} \int_a^{a+2^{i+1}t} w(y)^{p'} dy \\ &\geq C|I|^{(\frac{1}{p}-\alpha+1)p'} \sum_{i \geq 0} \int_{2^i t}^{2^{i+1}t} \frac{w^{p'}([a, a+s])}{s^{(2-\alpha)p'}} \frac{ds}{s} \\ &= C|I|^{(\frac{1}{p}-\alpha+1)p'} \int_t^\infty \frac{w^{p'}([a, a+s])}{s^{(2-\alpha)p'}} \frac{ds}{s}. \end{aligned}$$

In consequence,

$$\int_t^\infty \frac{w^{p'}([a, a+s])}{s^{(2-\alpha)p'}} \frac{ds}{s} \leq C \frac{w^{p'}([a, a+t])}{t^{(2-\alpha)p'}}.$$

Now, using Lemma 3.3 in [1] with $\varphi(s) = w^{p'}([a, a+s])$ y $r = (2-\alpha)p'$, there exist C and ϵ such that

$$\varphi(\theta t) \leq C\theta^{r-\epsilon}\varphi(t),$$

for every $t > 0$ and $\theta \geq 1$. That is,

$$w^{p'}([a, a+\theta t]) \leq C\theta^{(2-\alpha)p'-\epsilon}w^{p'}([a, a+t]),$$

for every $t > 0$ and $\theta \geq 1$. This completes the proof of (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (a) : Let $I = [a, a+t]$. If (b) holds, we have that

$$(4.2) \quad \left(\int_{a+t}^\infty \frac{w(y)^{p'}}{(y-a)^{(2-\alpha)p'}} dy \right)^{1/p'} = \left(\sum_{k=0}^\infty \int_{a+2^k t}^{a+2^{k+1}t} \frac{w(y)^{p'}}{(y-a)^{(2-\alpha)p'}} dy \right)^{1/p'}$$

$$\begin{aligned}
&\leq \left(\sum_{k=0}^{\infty} \frac{1}{(2^k t)^{(2-\alpha)p'}} w^{p'}([a+t, a+t+2^{k+1}t]) \right)^{1/p'} \\
&\leq C \left(\sum_{k=0}^{\infty} \frac{(2^{k+1})^{(2-\alpha)p'-\epsilon}}{(2^k t)^{(2-\alpha)p'}} w^{p'}([a+t, a+2t]) \right)^{1/p'} \\
&\leq C \left(\frac{1}{t} \int_{a+t}^{a+2t} w(y)^{p'} dy \right)^{1/p'} t^{\frac{1}{p'}-2+\alpha}.
\end{aligned}$$

Using the hypothesis $w \in RH^-(p')$ we obtain that (4.2) is bounded by

$$C \frac{1}{t} \int_a^{a+t} w(y) dy t^{\frac{1}{p'}-2+\alpha} = C \frac{w([a, a+t])}{t^{\frac{1}{p}+2-\alpha}},$$

which shows that $w \in H^-(\alpha, p)$.

The proof of $(b) \Rightarrow (c)$ is very simple and we shall omit it.

$(c) \Rightarrow (b)$: Taking $\theta = 1$ in (c) we have that $w \in RH^-(p')$. Using (c) and Hölder's inequality,

$$\begin{aligned}
\left(\frac{w^{p'}([a-t, a+\theta t])}{\theta t} \right)^{1/p'} &= \left(\frac{w^{p'}([a-t, a])}{\theta t} + \frac{w^{p'}([a, a+\theta t])}{\theta t} \right)^{1/p'} \\
&\leq \left(\frac{w^{p'}([a-t, a])}{\theta t} \right)^{1/p'} + C \theta^{\frac{1}{p}+1-\alpha-\frac{\epsilon}{p'}} \left(\frac{w^{p'}([a-t, a])}{t} \right)^{1/p'}.
\end{aligned}$$

We can suppose that $\frac{1}{p}+1-\alpha-\frac{\epsilon}{p'} > 0$, then taking into account that $\theta \geq 1$

$$\begin{aligned}
\left(\frac{w^{p'}([a-t, a-t+\theta t])}{\theta t} \right)^{1/p'} &\leq \left(\frac{w^{p'}([a-t, a+\theta t])}{\theta t} \right)^{1/p'} \\
&\leq C \theta^{\frac{1}{p}+1-\alpha-\frac{\epsilon}{p'}} \left(\frac{w^{p'}([a-t, a])}{t} \right)^{1/p'}.
\end{aligned}$$

From these inequalities with $a = b+t$ we obtain that

$$w^{p'}([b, b+\theta t]) \leq C \theta^{(2-\alpha)p'-\epsilon} w^{p'}([b, b+t]),$$

which completes the proof. $\square$

Remark 4.3. It is easy to see that if $w^{p'}$ belongs to A_1^- then, $w \in H^-(\alpha, p)$. On the other hand, applying Lemma 4.2 $(b) \Rightarrow (a)$, it follows that if $w(x) = |x|^\gamma$ with $0 < \gamma < 1/p - \alpha + 1$, then w belongs to $H^-(\alpha, p)$, but w does not belong to A_1^- . For $0 < \alpha < 1/p$, as an immediate consequence of Lemma 4.2 $(c) \Rightarrow (a)$ it follows that if $w^{p'}$ belongs to $A_{p'+1}^-$ then, w belongs to $H^-(\alpha, p)$.

The next two lemmas show that if w belongs to $H^-(\alpha, p)$, $1 < p < \infty$, then there exists $\eta > 0$ such that w belongs to $H^-(\alpha, q)$ for every $q : p - \eta < q < p + \eta$.

Lemma 4.4. *Let $1 < p < \infty$ and $w \in H^-(\alpha, p)$. Then, there exists $\delta_0 \in (0, 1)$ such that $w \in H^-(\alpha, (p'\delta)')$ for any $\delta : \delta_0 < \delta \leq 1$.*

Proof. It is a simple variant of Lemma 3.13 in [1]. $\square$

Lemma 4.5. *Let $1 < p < \infty$ and $w \in H^-(\alpha, p)$. Then, there exists $\tau_0 > 1$ such that $w \in H^-(\alpha, (p'\tau)')$ for any $1 \leq \tau \leq \tau_0$.*

Proof. Since $w \in RH^-(p')$ applying Theorem 5.3 in [9], there exists $\tau_0 > 1$ such that for every $\tau : 1 \leq \tau \leq \tau_0$ there exists a constant C such that

$$\left(\frac{1}{c-b} \int_b^c w^{p'\tau} \right)^{\frac{1}{p'\tau}} \leq C \left(\frac{1}{b-a} \int_a^b w \right) \leq C \left(\frac{1}{b-a} \int_a^b w^{p'} \right)^{\frac{1}{p'}},$$

for every $a < b < c$ with $c - b = 2(b - a)$. Let $I = [a, b]$. Using (4.3) we have that,

$$\begin{aligned} (4.4) \quad & \int_b^\infty \frac{w(y)^{p'\tau}}{(y-a)^{(2-\alpha)p'\tau}} dy = \sum_{k \geq 0} \int_{2^k|I| \leq y-a \leq 2^{k+1}|I|} \frac{w(y)^{p'\tau}}{(y-a)^{(2-\alpha)p'\tau}} dy \\ & \leq \sum_{k \geq 0} \frac{1}{(2^k|I|)^{(2-\alpha)p'\tau}} \int_{2^k|I| \leq y-a \leq 2^{k+1}|I|} w(y)^{p'\tau} dy \\ & \leq C \sum_{k \geq 0} \frac{1}{(2^k|I|)^{(2-\alpha)p'\tau-1}} \left(\frac{1}{2^k|I|} \int_{2^{k-1}|I| \leq y-a \leq 2^k|I|} w(y)^{p'} dy \right)^\tau \end{aligned}$$

Taking into account that $\tau > 1$, (4.4) is bounded by

$$\begin{aligned} & C \sum_{k \geq 0} 2^k|I| \left(\frac{1}{2^k|I|} \int_{2^{k-1}|I| \leq y-a \leq 2^k|I|} \frac{w(y)^{p'}}{(y-a)^{(2-\alpha)p'}} dy \right)^\tau \\ & \leq C|I|^{1-\tau} \left(\int_{\frac{|I|}{2} \leq y-a} \frac{w(y)^{p'}}{(y-a)^{(2-\alpha)p'}} dy \right)^\tau. \end{aligned}$$

Keeping in mind that $w \in H^-(\alpha, p)$ we have,

$$\begin{aligned} & \int_b^\infty \frac{w(y)^{p'\tau}}{(y-a)^{(2-\alpha)p'\tau}} dy \leq C|I|^{1-\tau} \left(\frac{w([a, a + |I|/2])}{|I|} |I|^{-1/p+\alpha-1} \right)^{p'\tau} \\ & = C \left(\frac{w(I)}{|I|} \frac{1}{|I|^{\frac{1}{(p'\tau)'}-\alpha+1}} \right)^{p'\tau}, \end{aligned}$$

which implies that $w \in H^-(\alpha, (p'\tau)')$. $\square$

Lemma 4.6. *Let $1 < p_1 < p_2 < \infty$. Suppose that $w \in H^-(\alpha, p_i)$ for $i = 1, 2$. Then $w \in H^-(\alpha, p)$ for every $p : p_1 < p < p_2$.*

Proof. This is an one-sided version of Lemma 3.15 in [1]. $\square$

Lemma 4.7. *Let $1 < p < \infty$ and $w \in RH^-(p')$. There exists a constant C such that for every $f \in \widetilde{L}_w^p$ and every bounded interval $I = [a, b]$, if we denote $\widetilde{I}^- = [a - \frac{|I|}{2}, a]$ then,*

$$\int_I |f(x)| dx \leq C \frac{w(\widetilde{I}^-)}{|I|^{1/p}} [f]_{p,w}.$$

Proof. Since $w \in RH^-(p')$ by Theorema 5.3 in [9], there exists $s > p'$ such that $w \in RH^-(s)$, that is, there exists a constant C such that for every bounded interval I ,

$$\left(\frac{1}{|I|} \int_I w(x)^s dx \right)^{1/s} \leq C \frac{w(\widetilde{I}^-)}{|I|}.$$

From this fact, the proof follows as in Lemma 4.1 of [1]. $\square$

Lemma 4.8. *Let $1 < p < \infty$ and $w \in H^-(\alpha, p)$. Then there exists a constant C such that for every $f \in \widetilde{L}_w^p$ and every bounded interval $I = [a, b]$,*

$$\int_b^\infty \frac{|f(y)|}{(y-a)^{2-\alpha}} dy \leq C \frac{w(I)}{|I|^{2+\frac{1}{p}-\alpha}} [f]_{p,w}.$$

Proof. Taking into account Lemma 4.4 and Lemma 4.5, the proof of this lemma is similar to one in Lemma 4.4 of [1]. $\square$

Lemma 4.9. *Let $\alpha > 0$ and $\delta \geq 0$ such that $0 < \alpha + \delta < 1$. Let $w \in D^-$. For $a < b$, we denote $c = \frac{a+b}{2}$ and $I = [c, b]$. Then, for every $f \in \mathcal{L}_w(\delta)$, there exists a constant C such that,*

(i)

$$\int_b^\infty \frac{|f(y) - f_I|}{(y-a)^{2-\alpha}} dy \leq C \|f\|_{\mathcal{L}_w(\delta)} \int_c^\infty \frac{w(y)}{(y-a)^{2-\alpha-\delta}} dy.$$

(ii)

$$\int_a^b \frac{|f(y) - f_I|}{(y-a)^{1-\alpha}} dy \leq C \|f\|_{\mathcal{L}_w(\delta)} \int_a^c \frac{w(y)}{(y-a)^{1-\alpha-\delta}} dy.$$

Proof. The proof of (i) and (ii) are similar, then we only prove part (i).

For every $j \geq 0$, let $I_j = [a + 2^j|I|, a + 2^{j+1}|I|]$. We observe that $I_0 = [a + |I|, a + 2|I|] = [c, b] = I$. Since $f \in \mathcal{L}_w(\delta)$ we have that,

$$(4.5) \quad \int_b^\infty \frac{|f(y) - f_I|}{(y-a)^{2-\alpha}} dy = \sum_{j=1}^\infty \int_{a+2^j|I|}^{a+2^{j+1}|I|} \frac{|f(y) - f_I|}{(y-a)^{2-\alpha}} dy$$

$$\begin{aligned}
&\leq \sum_{j=1}^{\infty} \frac{1}{(2^j |I|)^{2-\alpha}} \int_{a+2^j |I|}^{a+2^{j+1} |I|} |f(y) - f_{I_0}| dy \\
&\leq \sum_{j=1}^{\infty} \frac{1}{(2^j |I|)^{2-\alpha}} \left[\int_{a+2^j |I|}^{a+2^{j+1} |I|} |f(y) - f_{I_j}| dy + 2^j |I| \sum_{k=1}^j |f_{I_k} - f_{I_{k-1}}| \right] \\
&\leq \sum_{j=1}^{\infty} \frac{1}{(2^j |I|)^{1-\alpha}} \left[C \|f\|_{\mathcal{L}_w(\delta)} w(I_j) (2^j |I|)^{\delta-1} + \sum_{k=1}^j \frac{1}{|I_{k-1}|} \int_{I_{k-1}} |f(y) - f_{I_k}| dy \right].
\end{aligned}$$

Using that $f \in \mathcal{L}_w(\delta)$ and $w \in D^-$ we obtain the estimate,

$$\frac{1}{|I_{k-1}|} \int_{I_{k-1}} |f(y) - f_{I_k}| dy \leq C \|f\|_{\mathcal{L}_w(\delta)} w(I_{k-1}) (2^{k-1} |I|)^{\delta-1}.$$

Then applying Fubini's Theorem, (4.5) is bounded by

$$\begin{aligned}
&C \|f\|_{\mathcal{L}_w(\delta)} \sum_{j=1}^{\infty} \frac{1}{(2^j |I|)^{1-\alpha}} \sum_{k=0}^j w(I_k) (2^k |I|)^{\delta-1} \\
&= C \|f\|_{\mathcal{L}_w(\delta)} \sum_{k=0}^{\infty} w(I_k) (2^k |I|)^{\delta-1} \sum_{j=k}^{\infty} \frac{1}{(2^j |I|)^{1-\alpha}} \\
&= C \|f\|_{\mathcal{L}_w(\delta)} \sum_{k=0}^{\infty} \frac{1}{(2^k |I|)^{2-\alpha-\delta}} \int_{a+2^k |I|}^{a+2^{k+1} |I|} w(y) dy \\
&\leq C \|f\|_{\mathcal{L}_w(\delta)} \int_c^{\infty} \frac{w(y)}{(y-a)^{2-\alpha-\delta}} dy,
\end{aligned}$$

as we wanted to prove. $\square$

5. PROOF OF THE THEOREMS 2.3 AND 2.4

Proof of the Theorem 2.3.

(i) $\Rightarrow$ (ii) : Let $w \in H^-(\alpha, p)$ and $x_0 \in \mathbb{R}$. Given $f \in \widetilde{L}_w^p$ let $\widetilde{I}_\alpha^+(f)$ define as in (2.2). Choose a bounded interval $I = [a, a+h]$. We consider $I_0 = [a+2h, x_0]$ if $a+2h \leq x_0$ and $I_0 = \emptyset$ if $x_0 < a+2h$, and we also define $I_1 = [x_0, a+2h]$ if $x_0 < a+2h$ and $I_1 = \emptyset$ in the other case. We set

$$a_I = \int_{I_0} \frac{f(y)}{(y-a)^{1-\alpha}} dy + \int_{x_0}^{\infty} \left[\frac{1 - \chi_{I_1}(y)}{(y-a)^{1-\alpha}} - \frac{1 - \chi_{[x_0, x_0+1]}(y)}{(y-x_0)^{1-\alpha}} \right] f(y) dy.$$

We shall show that a_I is a finite constant.

Suppose that $x_0 < a+2h$. Let n be a positive integer such that $a+2^n h > x_0+1$ and $|a-x_0| \leq 2^{n-1} h$. Then,

$$a_I = \left(\int_{x_0}^{a+2^n h} + \int_{a+2^n h}^{\infty} \right) \left[\frac{1 - \chi_{[x_0, a+2h]}(y)}{(y-a)^{1-\alpha}} - \frac{1 - \chi_{[x_0, x_0+1]}(y)}{(y-x_0)^{1-\alpha}} \right] f(y) dy$$

$$= J_1 + J_2.$$

For each $y \geq a + 2^n h$, by Mean Value Theorem, there exists $\theta : 0 < \theta < 1$ such that,

$$\left| \frac{1}{(y-a)^{1-\alpha}} - \frac{1}{(y-x_0)^{1-\alpha}} \right| \leq C \frac{|x_0-a|}{|y-\theta a-(1-\theta)x_0|^{2-\alpha}} \leq C \frac{|x_0-a|}{|y-a|^{2-\alpha}}.$$

Then, applying Lemma 4.8, we have that

$$|J_2| \leq C|x_0-a| \int_{a+2^n h}^{\infty} \frac{|f(y)|}{|y-a|^{2-\alpha}} dy \leq C|x_0-a| \frac{w([a, a+2^n h])}{(2^n h)^{2+\frac{1}{p}-\alpha}} [f]_{p,w} < \infty.$$

On the other hand, since $f \in \widetilde{L}_w^p$ and using Lemma 4.7, we get

$$\begin{aligned} |J_1| &\leq \int_{a+2h}^{a+2^n h} \frac{|f(y)|}{(y-a)^{1-\alpha}} dy + \int_{x_0+1}^{a+2^n h} \frac{|f(y)|}{(y-x_0)^{1-\alpha}} dy \\ &\leq \frac{1}{(2h)^{1-\alpha}} \int_{a+2h}^{a+2^n h} |f(y)| dy + \int_{x_0+1}^{a+2^n h} |f(y)| dy < \infty. \end{aligned}$$

The case $x_0 \geq a + 2h$ can be proved in a similar way.

Now, let

$$\begin{aligned} (5.1) \quad A(x) &= \int_x^{a+2h} \frac{f(y)}{(y-x)^{1-\alpha}} dy + \int_{a+2h}^{\infty} \left[\frac{1}{(y-x)^{1-\alpha}} - \frac{1}{(y-a)^{1-\alpha}} \right] f(y) dy \\ &= A_1(x) + A_2(x). \end{aligned}$$

It follows that,

$$(5.2) \quad \widetilde{I}_\alpha^+(f)(x) = A(x) + a_I.$$

We shall show that,

$$\int_I |\widetilde{I}_\alpha^+(f)(x) - a_I| dx \leq C|I|^{\alpha-1/p} w(I^-) [f]_{p,w}.$$

We observe that taking into account (5.2) and (5.1) it is sufficient to prove that

$$\int_I |A_j(x)| dx \leq C|I|^{\alpha-1/p} w(I^-) [f]_{p,w},$$

for $j = 1, 2$. Applying Mean Value Theorem, Lemma 4.8 and Lemma 4.1 part (iii) for every $x \in I = [a, a+h]$ we have that,

$$|A_2(x)| \leq \int_{a+2h}^{\infty} \left| \frac{1}{(y-x)^{1-\alpha}} - \frac{1}{(y-a)^{1-\alpha}} \right| |f(y)| dy$$

$$\begin{aligned}
&\leq Ch \int_{a+2h}^{\infty} \frac{|f(y)|}{|y-a|^{2-\alpha}} dy \leq Ch \frac{w([a, a+2h])}{(2h)^{2+\frac{1}{p}-\alpha}} [f]_{p,w} \\
&\leq C \frac{w([a-h, a])}{h^{1+\frac{1}{p}-\alpha}} [f]_{p,w}.
\end{aligned}$$

Therefore,

$$\int_I |A_2(x)| dx \leq C |I|^{\alpha-1/p} w(I^-) [f]_{p,w}.$$

With respect to $A_1(x)$, changing the order of integration and applying Lemma 4.7,

$$\begin{aligned}
&\int_a^{a+h} |A_1(x)| dx \leq \int_a^{a+h} \int_x^{a+2h} \frac{|f(y)|}{(y-x)^{1-\alpha}} dy dx \\
&\leq \int_a^{a+2h} |f(y)| \int_a^y \frac{dx}{(y-x)^{1-\alpha}} dy \leq Ch^\alpha \int_a^{a+2h} |f(y)| dy \\
&\leq Ch^{\alpha-1/p} w([a-h, a]) [f]_{p,w},
\end{aligned}$$

which completes the proof of (i) $\Rightarrow$ (ii).

The implication (ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i) Let $a \in \mathbb{R}$ and $h > 0$. We consider $f \geq 0$ such that $\text{sop}(f) \subseteq [a+4h, \infty)$. For each $x \in [a, a+h]$ we have that,

$$\begin{aligned}
&|I_\alpha^+(f)(x) - I_\alpha^+(f)_{[a+2h, a+3h]}| \\
&= \frac{1}{h} \int_{a+2h}^{a+3h} \int_{a+4h}^{\infty} f(y) \left[\frac{1}{(y-t)^{1-\alpha}} - \frac{1}{(y-x)^{1-\alpha}} \right] dy dt.
\end{aligned}$$

Applying Mean Value Theorem, for each $y \geq a+4h$ we obtain,

$$\frac{1}{(y-t)^{1-\alpha}} - \frac{1}{(y-x)^{1-\alpha}} \geq C \frac{|x-t|}{(y-a)^{2-\alpha}} \geq C \frac{h}{(y-a)^{2-\alpha}}.$$

In consequence,

$$|I_\alpha^+(f)(x) - I_\alpha^+(f)_{[a+2h, a+3h]}| \geq Ch \int_{a+4h}^{\infty} \frac{f(y)}{(y-a)^{2-\alpha}} dy.$$

Then, if $f \in L_w^p$, using (iii) we have that,

$$\begin{aligned}
Ch^2 \int_{a+4h}^{\infty} \frac{f(y)}{(y-a)^{2-\alpha}} dy &\leq 2 \int_a^{a+3h} |I_\alpha^+(f)(x) - I_\alpha^+(f)_{[a, a+3h]}| dx \\
&\leq C(3h)^\beta w([a-3h, a]) \left[\int \left(\frac{f(y)}{w(y)} \right)^p dy \right]^{1/p}.
\end{aligned}$$

Now, taking into account that $\beta = \alpha - 1/p$ it follows that,

$$(5.3) \quad h^{1/p-\alpha+1} \int_{a+4h}^{\infty} \frac{f(y)}{(y-a+3h)^{2-\alpha}} dy \\ \leq C \frac{w([a-3h, a+4h])}{h} \left[\int_{a+4h}^{\infty} \left(\frac{f(y)}{w(y)} \right)^p dy \right]^{1/p}.$$

For each $m > 2$ we put,

$$f_m(y) = \frac{w(y)^{p'}}{(y-a+3h)^{\frac{2-\alpha}{p-1}}} \chi_{[a+4h, a+2^m h]}(y) \chi_{\{0 \leq w \leq m\}}(y).$$

It is easy to check that $f_m \in L_w^p$. Using (5.3) with f_m and taking the limit, we obtain that

$$h^{1/p-\alpha+1} \left(\int_{a+4h}^{\infty} \frac{w(y)^{p'}}{(y-a+3h)^{(2-\alpha)p'}} dy \right)^{1/p'} \leq C \frac{w([a-3h, a+4h])}{h},$$

which shows that $w \in H^-(\alpha, p)$. $\square$

Remark 5.1. By Theorem 2.1, if $0 \leq \beta < 1$, we can substitute in Theorem 2.3, $\mathcal{L}_w(\beta)$ for $\mathcal{L}_w^-(\beta)$. That is not possible for $-1 < \beta < 0$. In fact, if w and f are defined as in Remark 3.2 part (ii), then

$$I_{\alpha}^+(f)(x) = \begin{cases} \frac{\Gamma(\alpha)}{a^{\alpha}} e^{-ax} & , \quad x \geq 0 \\ \frac{|x|^{\alpha}}{\alpha} + \frac{e^{-ax}}{a^{\alpha}} \int_{a|x|}^{\infty} e^{-u} u^{\alpha-1} du & , \quad x < 0. \end{cases}$$

Therefore, the same arguments used in Remark 3.2 imply that $I_{\alpha}^+(f) \notin \mathcal{L}_w(\beta)$.

Proof of Theorem 2.4.

(i) $\Rightarrow$ (ii) Let $R > 0$. For any $a \in \mathbb{R}$, applying Fubini's Theorem and taking into account that w is a locally integrable function, we have that

$$\int_a^{a+1} \int_x^{x+R} \frac{w(y)}{(y-x)^{1-\alpha}} dy dx < \infty.$$

In consequence, for almost every x and every $R > 0$

$$(5.4) \quad \int_x^{x+R} \frac{w(y)}{(y-x)^{1-\alpha}} dy < \infty.$$

Let x_0 satisfying (5.4). We consider

$$(5.5) \quad \widetilde{I}_{\alpha}^+(f)(x) = \int_{-\infty}^{\infty} \left[\frac{\chi_{[x_0, \infty)}(y)}{|y-x_0|^{1-\alpha}} - \frac{\chi_{[x, \infty)}(y)}{|y-x|^{1-\alpha}} \right] f(y) dy.$$

We shall show that if $f \in \mathcal{L}_w(0)$ then $\widetilde{I}_{\alpha}^+(f)$, defined as in (5.5), is finite for every x satisfying (5.4). Fix x satisfying (5.4). Suppose that $x_0 < x$ and let $R \in \mathbb{Q} : x_0 < x \leq x_0 + R/4$. We consider the interval $I = [x_0 + R/2, x_0 +$

$R]$. Taking into account that the function $g(y) = \frac{\chi_{[x_0, \infty)}(y)}{|y-x_0|^{1-\alpha}} - \frac{\chi_{[x, \infty)}(y)}{|y-x|^{1-\alpha}}$ is integrable and $\int_{-\infty}^{\infty} g(y)dy = 0$ we can write,

$$\begin{aligned}\widetilde{I}_{\alpha}^{+}(f)(x) &= \int_{-\infty}^{\infty} \left[\frac{\chi_{[x_0, \infty)}(y)}{|y-x_0|^{1-\alpha}} - \frac{\chi_{[x, \infty)}(y)}{|y-x|^{1-\alpha}} \right] [f(y) - f_I] dy \\ &= I_1(x) + I_2(x),\end{aligned}$$

where,

$$I_1(x) = \int_{x_0}^{x_0+R} \quad \text{and} \quad I_2(x) = \int_{x_0+R}^{\infty}.$$

We shall prove

$$(5.6) \quad |\widetilde{I}_{\alpha}^{+}(f)(x)| \leq C\|f\|_{\mathcal{L}_w(0)} \left[\int_{x_0}^{x_0+5R/4} \frac{w(y)}{(y-x_0)^{1-\alpha}} dy + \int_x^{x+5R/4} \frac{w(y)}{(y-x)^{1-\alpha}} dy \right].$$

We observe that,

$$|I_1(x)| \leq \int_{x_0}^{x_0+R} \frac{|f(y) - f_I|}{|y-x_0|^{1-\alpha}} dy + \int_x^{x_0+R} \frac{|f(y) - f_I|}{|y-x|^{1-\alpha}} dy$$

Let $J = [x + R/2, x + R]$. Applying Lemma 4.9 part (ii) we have that

$$\begin{aligned}(5.7) \quad |I_1(x)| &\leq \int_{x_0}^{x_0+R} \frac{|f(y) - f_I|}{|y-x_0|^{1-\alpha}} dy \\ &\quad + \int_x^{x+R} \frac{|f(y) - f_J|}{|y-x|^{1-\alpha}} + |f_I - f_J| \int_x^{x+R} \frac{dy}{|y-x|^{1-\alpha}} \\ &\leq C\|f\|_{\mathcal{L}_w(0)} \int_{x_0}^{x_0+R/2} \frac{w(y)}{(y-x_0)^{1-\alpha}} dy + C\|f\|_{\mathcal{L}_w(0)} \int_x^{x+R/2} \frac{w(y)}{(y-x)^{1-\alpha}} dy \\ &\quad + \frac{R^{\alpha}}{\alpha} |f_I - f_J|.\end{aligned}$$

Since $x_0 < x < x_0 + R/4$ and $f \in \mathcal{L}_w(0)$ we have,

$$R^{\alpha} |f_I - f_J| \leq C\|f\|_{\mathcal{L}_w(0)} \int_{x_0}^{x_0+5/4R} \frac{w(y)}{(y-x_0)^{1-\alpha}} dy.$$

Then, by (5.7)

$$|I_1(x)| \leq C\|f\|_{\mathcal{L}_w(0)} \left[\int_{x_0}^{x_0+5R/4} \frac{w(y)}{(y-x_0)^{1-\alpha}} dy + \int_x^{x+5R/4} \frac{w(y)}{(y-x)^{1-\alpha}} dy \right].$$

Now, let us estimate I_2 . Applying Mean Value Theorem,

$$|I_2(x)| \leq \int_{x_0+R}^{\infty} \left| \frac{1}{|y-x_0|^{1-\alpha}} - \frac{1}{|y-x|^{1-\alpha}} \right| |f(y) - f_I| dy$$

$$\leq C_\alpha |x_0 - x| \int_{x_0+R}^{\infty} \frac{|f(y) - f_{[x_0+R/2, x_0+R]}|}{(y - x_0)^{2-\alpha}} dy.$$

Using Lemma 4.9 part (i) and taking into account that $w \in H^-(\alpha, \infty)$ we get,

$$\begin{aligned} |I_2(x)| &\leq C_\alpha R \|f\|_{\mathcal{L}_w(0)} \int_{x_0+R/2}^{\infty} \frac{w(y)}{(y - x_0)^{2-\alpha}} dy \\ &\leq C_\alpha R \|f\|_{\mathcal{L}_w(0)} \frac{w([x_0, x_0 + R/2])}{R^{2-\alpha}} \\ &\leq C_\alpha \|f\|_{\mathcal{L}_w(0)} \int_{x_0}^{x_0+R/2} \frac{w(y)}{(y - x_0)^{1-\alpha}} dy. \end{aligned}$$

Then, if $x_0 < x < x_0 + R/4$ or in the case $x_0 - R/4 < x < x_0$, we have that (5.6) holds. Since $\mathbb{R} = \cup_{R \in \mathcal{Q}_{>0}} [x_0 - R/4, x_0 + R/4]$, it follows that $\widetilde{I}_\alpha^+(f)(x)$ is finite for almost every x .

Let us show that $\widetilde{I}_\alpha^+(f) \in \mathcal{L}_w(\alpha)$. For almost every $x_1 < x_2$, if we define $R = 4|x_1 - x_2|$, we have that $x_1 < x_2 \leq x_1 + R/4$ and using (5.6) we get

$$\begin{aligned} &|\widetilde{I}_\alpha^+(f)(x_1) - \widetilde{I}_\alpha^+(f)(x_2)| \\ &\leq \int_{-\infty}^{\infty} \left| \frac{\chi_{[x_1, \infty)}(y)}{(y - x_1)^{1-\alpha}} - \frac{\chi_{[x_2, \infty)}(y)}{(y - x_2)^{1-\alpha}} \right| |f(y) - f_{[x_1+R/2, x_1+R]}| dy \\ &\leq C \|f\|_{\mathcal{L}_w(0)} \left[\int_{x_1}^{x_1+5|x_1-x_2|} \frac{w(y)}{(y - x_1)^{1-\alpha}} dy + \int_{x_2}^{x_2+5|x_1-x_2|} \frac{w(y)}{(y - x_2)^{1-\alpha}} dy \right]. \end{aligned}$$

Taking into account that $w \in D^-$ and using Proposition 3.3 it follows that $\widetilde{I}_\alpha^+(f) \in \mathcal{L}_w(\alpha)$.

(ii) $\Rightarrow$ (i) This implication is similar to (iii) $\Rightarrow$ (i) in Theorem 2.3. $\square$

Corollary 5.2. *Let $\alpha, \delta \in \mathbb{R}^+$ such that $0 < \alpha + \delta < 1$. The following statements are equivalent.*

- (a) $w \in H^-(\delta, \infty)$ and the operator I_α can be extended to a linear bounded operator $\widetilde{I}_\alpha^+ : \mathcal{L}_w(\delta) \longrightarrow \mathcal{L}_w(\alpha + \delta)$.
- (b) $w \in H^-(\alpha + \delta, \infty)$.

Proof. The proof is a simple variant of Corollary 2.12 in [1]. $\square$

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