An interpolation theorem between one-sided Hardy spaces

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Abstract

In this paper we generalize an interpolation result due to J.-O. Strömberg and A. Torchinsky ([6]) to the case of one-sided Hardy spaces. This generalization is important in the study of the weak type (1,1) for lateral strongly singular operators ([1]). We shall need an atomic decomposition in which for every atom there exists another supported contiguously at its right. In order to obtain this decomposition we have developed a rather simple technique to break up an atom into a sum of others atoms.

1 Definitions and prerequisites

Let $f(x)$ be a Lebesgue measurable function defined on $\mathbb{R}$. The one-sided Hardy-Littlewood maximal functions $M^+f(x)$ and $M^-f(x)$ are defined as

$$M^+f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f(t)|\, dt \text{ and } M^-f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f(t)|\, dt.$$

As usual, a weight ω is a measurable and non-negative function. If $E \subset \mathbb{R}$ is a Lebesgue measurable set, we denote its ω-measure by $\omega(E) = \int_E \omega(t)dt$. A function $f(x)$ belongs to L^s_ω, $0 < s < \infty$, if $\|f\|_{L^s_\omega} = \left(\int_{-\infty}^{\infty} |f(x)|^s \omega(x)dx\right)^{1/s}$ is finite.

A weight ω belongs to the class A^+_s, $1 < s < \infty$, defined by E. Sawyer in [5], if there exists a constant $C_{\omega,s}$ such that if $-\infty < a < b < c < \infty$ then

$$\left(\int_a^b \omega(t)dt\right)\left(\int_b^c \omega(t)^{-\frac{1}{s-1}} dt\right)^{s-1} \leq C_{\omega,s} \,(c-a)^s\,,$$

In the limit case of $s = 1$ we say that ω belongs to the class A^+_1 if, and only if, $M^-\omega(x) \leq C_{\omega,1}\omega(x)$ a.e. We also consider the class $A^+_\infty = \bigcup_{s\geq 1} A^+_s$.

In general a weight belonging to the A^+_∞ can be equal to zero in a set with positive measure. For simplicity in this paper we shall assume all weights being positive almost everywhere.

As usual, $C^\infty_0(\mathbb{R})$ denotes the set of all functions with compact support having derivatives of all orders. We denote by $\mathcal{D}$ the space of all functions in

$C_0^\infty(\mathbb{R})$ equipped with the usual topology and by $\mathcal{D}'$ the space of distributions on $\mathbb{R}$.

Given a positive integer γ and $x \in \mathbb{R}$, we shall say a function ψ in $C_0^\infty(\mathbb{R})$, belongs to the class $\Phi_\gamma(x)$ if there exists a bounded interval $I_\psi = [x, b]$ containing the support of ψ such that $D^\gamma \psi$ satisfies

$$|I_\psi|^{\gamma+1} \left\| D^\gamma \psi \right\|_\infty \leq 1.$$

For $f \in \mathcal{D}'$ we consider the one-sided maximal function

$$f^*_{+,\gamma}(x) = \sup \left\{ |\langle f, \psi \rangle| : \psi \in \Phi_\gamma(x) \right\}.$$

Let $\omega \in A_s^+$, $0 < p < \infty$ and γ a positive integer such that $p(\gamma+1) > s$. We say that a distribution f in $\mathcal{D}'$ belongs to $H_+^p(\omega)$ if the "p-norm" $\|f\|_{H_+^p(\omega)} = \left(\int_{-\infty}^\infty f^*_{+,\gamma}(x)^p \omega(x) dx \right)^{1/p}$ is finite. These spaces have been defined by L. de Rosa and C. Segovia in [3]. They also prove in [4] that definition does not depend on γ.

Let N be an integer. A function $a(x)$ defined on $\mathbb{R}$ is called a (∞, N)-atom if there exists an interval I containing the support of $a(x)$, such that

(i) $\|a\|_\infty \leq 1$.

(ii) The identity $\int_I a(y) y^k dy = 0$ holds for every integer k, $0 \leq k \leq N$.

The following theorem gives an atomic decomposition of the $H_+^p(\omega)$ spaces:

Theorem 1 *Let $\omega \in A_s^+$ and $0 < p < \infty$. Then there is an integer $N(p, \omega)$ with the following property: given any $f \in H_+^p(\omega)$ and $N \geq N(p, \omega)$, we can find a sequence $\{\lambda_k\}$ of positive coefficients and a sequence $\{a_k\}$ of (∞, N)-atoms with support contained in intervals $\{I_k\}$ respectively, such that the sum $\sum_k \lambda_k a_k$ converges unconditionally to f both in the sense of distributions and in the $H_+^p(\omega)$-norm. Moreover*

$$C_1 \left\| \sum \lambda_k \chi_{I_k} \right\|_{L^p_\omega} \leq \|f\|_{H_+^p(\omega)},$$

holds with $C_1 > 0$ not depending on f.
Conversely, if we have a sequence $\{\lambda_k\}$ of positive coefficients and a sequence $\{a_k\}$ of (∞, N)-atoms with support contained in intervals $\{I_k\}$ respectively, such that $\|\sum \lambda_k \chi_{I_k}\|_{L^p_\omega} < \infty$, then $\sum_k \lambda_k a_k$ converges unconditionally both in the sense of distributions and in the $H_+^p(\omega)$-norm to an element $f \in H_+^p(\omega)$ and

$$\|f\|_{H_+^p(\omega)} \leq C_2 \left\| \sum \lambda_k \chi_{I_k} \right\|_{L^p_\omega},$$

holds with C_2 not depending on f.
Moreover, in case $f \in L^1_{loc}(\mathbb{R})$ then

$$\left| \sum \lambda_k^r \chi_{I_k}(x) \right| \leq C_r f^*_{+,N}(x)^r, \tag{1}$$

holds for every $r > 0$ with $C_r = c^r \frac{2^r}{2^r - 1}$ where c is an absolute constant.

For $0 < p \leq 1$, this result is essentially proved in [3], and it can be generalized to the case $p > 1$ following the ideas of Theorem 1 of Chapter VII in [6].

Trough this paper the letters c and C will mean positive finite constants not necessarily the same at each occurrence.

2 Statement of the main results

We shall denote by Ω the strip in the complex plane $\{z \in \mathbb{C} : 0 \leq \mathrm{Re}(z) \leq 1\}$.

For $0 < p_0, p_1 < \infty$ and $z \in \Omega$ we define $p(z)$ as

$$\frac{1}{p(z)} = \frac{1-z}{p_0} + \frac{z}{p_1}.$$

Let ω and ν be weights in A_∞^+ such that $\omega(x) > 0$ and $\nu(x) > 0$ for almost $x \in \mathbb{R}$.

Given $0 \leq u \leq 1$ we define

$$\mu(u) = \frac{u\, p(u)}{p_1},$$

and, using the same notation,

$$\mu(u)(x) = \omega(x)^{1-\mu(u)} \nu(x)^{\mu(u)}. \tag{2}$$

It will be proved in Lemma 4 that $\mu(u) \in A_\infty^+$.

In the same way, we define $q(z)$ and $\tilde{\mu}(u)$, for $0 < q_0, q_1 < \infty$, $\tilde{\omega}$ and $\tilde{\nu}$ weights in A_∞^+.

Let s be a fixed real number, $0 < s < 1$, we shall put $p = p(s)$, $q = q(s)$, $\mu = \mu(s)$, $\tilde{\mu} = \tilde{\mu}(s)$.

With this notation we have:

Theorem 2 *Let $T_z : H_+^{p_0}(\omega) + H_+^{p_1}(\nu) \longrightarrow H_+^{q_0}(\tilde{\omega}) + H_+^{q_1}(\tilde{\nu})$ be a family of linear operators for $z \in \Omega$ such that if $f \in H_+^{p_0}(\omega) + H_+^{p_1}(\nu)$ then $T_z f * \phi_t(x)$ is uniformly continuous and bounded for $z \in \Omega$ and (x,t) in any compact subset of $\{(x,t) : x \in \mathbb{R},\, t > 0\}$ and analytic for z in the interior of Ω.*
If, in addition, for some constants $k, \beta > 0$,

$$\|T_{it} f\|_{H_+^{q_0}(\tilde{\omega})} \leq k e^{\beta|t|} \|f\|_{H_+^{p_0}(\omega)} \ \text{ and } \ \|T_{1+it} f\|_{H_+^{q_1}(\tilde{\nu})} \leq k e^{\beta|t|} \|f\|_{H_+^{p_1}(\nu)}$$

then there exists a constant C such that

$$\|T_s f\|_{H_+^{q}(\tilde{\mu})} \leq C \|f\|_{H_+^{p}(\mu)},$$

for all $f \in H_+^p(\mu)$.

By a standar argument, this theorem is a consequence of the following result:

Theorem 3 *Let $\phi \in C_0^\infty(\mathbb{R})$ with $supp(\phi) \subset (-\infty, 0]$ and $\int \phi \neq 0$.*
1) Let $f(z)$ be a function of the complex variable $z \in \Omega$ which takes values in $H_+^{p_0}(\omega) + H_+^{p_1}(\nu)$, such that $F(x,t,z) = f(z) * \phi_t(x)$ is uniformly continuous and bounded for $z \in \Omega$ and (x,t) in any compact subset of $\{(x,t) : x \in \mathbb{R},\, t > 0\}$ and analytic for z in the interior of Ω. If $f(it) \in H_+^{p_0}(\omega)$, $sup_t \|f(it)\|_{H_+^{p_0}(\omega)} < \infty$, $f(1+it) \in H_+^{p_1}(\nu)$ and $sup_t \|f(1+it)\|_{H_+^{p_1}(\nu)} < \infty$ then $f(s) \in H_+^p(\mu)$ and

$$\|f(s)\|_{H_+^p(\mu)} \leq c \left(sup_t \, \|f(it)\|_{H_+^{p_0}(\omega)} \right)^{1-s} \left(sup_t \, \|f(1+it)\|_{H_+^{p_1}(\nu)} \right)^s .$$

2) If $f \in H_+^p(\mu)$, there exists a function $f(z)$ such that $F(x,t,z)$ has the properties stated in the first part, $f(s) = f$ and

$$\|f(u+it)\|_{H_+^{p(u)}(\mu(u))}^{p(u)} \leq c \, \|f\|_{H_+^p(\mu)}^p \tag{3}$$

holds for $z = u + it \in \Omega$ and c does not depend on f.

3 Some preliminary results

In the following lemmas we state some results we shall need to prove Theorem 3. They correspond to Lemma 7 and Lemma 8 of Chapter XII of [6].

Lemma 4 *Let $\omega \in A_p^+$, $\nu \in A_q^+$, $0 < \delta < 1$ and $\eta > 0$.*
If $s = p(1-\delta) + q\delta$ then

$$\omega(x)^{1-\delta} \nu(x)^\delta \in A_s^+,$$

and there exists a constant $C = C(\eta, \delta, \omega, \nu)$ such that

$$\left(\frac{1}{|I^-|} \int_{I^-} \omega(x) dx \right)^{1-\delta} \left(\frac{1}{|I^-|} \int_{I^-} \nu(x) dx \right)^\delta \leq C \left(\frac{1}{|I^+|} \int_{I^+} \omega(x)^{1-\delta} \nu(x)^\delta dx \right),$$

holds for every pair of intervals $I^- = (a,b)$ and $I^+ = (b,c)$ with $b - a = \eta\,(c-b)$.

Proof. Assume $p > 1$, $q > 1$. By applying Hölder's inequality with exponents $\alpha = \frac{s-1}{(p-1)(1-\delta)}$ and $\beta = \frac{s-1}{(q-1)\delta}$, it is easy to see that $\omega(x)^{1-\delta}\nu(x)^\delta \in A_s^+$ with constant $C_{\omega,p}^{1-\delta} C_{\nu,q}^\delta$.

Let us prove the rest of the lemma. Since $\omega \in A_p^+$, $\nu \in A_q^+$ and by Hölder's inequality with exponents α and β, we have

$$\begin{aligned}
&\left(\int_{I^-} \omega(x) dx \right)^{1-\delta} \left(\int_{I^-} \nu(x) dx \right)^\delta \\
&\leq C_{\omega,p}^{1-\delta} C_{\nu,q}^\delta \,(c-a)^s \left[\left(\int_{I^+} \omega(x)^{-\frac{1}{p-1}} dx \right)^{(p-1)(1-\delta)} \left(\int_{I^+} \nu(x)^{-\frac{1}{q-1}} dx \right)^{(q-1)\delta} \right]^{-1} \\
&\leq C_{\omega,p}^{1-\delta} C_{\nu,q}^\delta \,(c-a)^s \left(\int_{I^+} \left(\omega(x)^{1-\delta} \nu(x)^\delta \right)^{-\frac{1}{s-1}} dx \right)^{(s-1)(-1)} \\
&\leq C_{\omega,p}^{1-\delta} C_{\nu,q}^\delta \left(\frac{c-a}{|I^+|} \right)^s \int_{I^+} \omega(x)^{1-\delta} \nu(x)^\delta dx.
\end{aligned}$$

If $b-a=\eta(c-b)$ then

$$\left(\frac{1}{|I^{-}|}\int_{I^{-}}\omega(x)dx\right)^{1-\delta}\left(\frac{1}{|I^{-}|}\int_{I^{-}}\nu(x)dx\right)^{\delta}\leq C\frac{1}{|I^{+}|}\int_{I^{+}}\omega(x)^{1-\delta}\nu(x)^{\delta}dx,$$

where $C=C_{\omega,p}^{1-\delta}C_{\nu,q}^{\delta}\frac{(1+\eta)^{s}}{\eta}$. ∎

Lemma 5 *Given $0<p<\infty$, $\eta>0$ there exists a constant $C=C(p,\eta)$ such that if δ is a weight on the real line then*

$$\left\|\sum\lambda_k\chi_{I_k}\right\|_{L^p_\delta}\leq C\left\|\sum\lambda_k\chi_{E_k}\right\|_{L^p_\delta}$$

holds for every $\lambda_k>0$ and for all intervals I_k and all $\delta-$measurable E_k, $E_k\subset I_k$ with $\delta(E_k)\geq\eta\delta(I_k)$.

Proof. The main ideas of the proof can be found in page 116 of [6]. For the sake of completeness we give a proof here.

For the case $1\leq p<\infty$, let $g\in L^{p'}_\delta$, where $pp'=p+p'$. We assume $g\geq 0$ and $\|g\|_{L^{p'}_\delta}=1$. Since $M_\delta(g)(y)=\sup_{I:\ y\in I}\frac{1}{\delta(I)}\int_I g(t)\,\delta(t)\,dt$ then we have,

$$\int\chi_{I_k}(y)g(y)\delta(y)dy\leq\delta(I_k)\inf_{y\in E_k}M_\delta(g)(y)$$
$$\leq\eta^{-1}\delta(E_k)\inf_{y\in E_k}M_\delta(g)(y)\leq\eta^{-1}\int\chi_{E_k}(y)M_\delta(g)(y)\delta(y)dy.$$

Thus,

$$\int\left(\sum\lambda_k\chi_{I_k}(y)\right)g(y)\delta(y)dy\leq\eta^{-1}\int\left(\sum\lambda_k\chi_{E_k}(y)\right)M_\delta(g)(y)\delta(y)dy$$
$$\leq\eta^{-1}\left\|\sum\lambda_k\chi_{E_k}\right\|_{L^p_\delta}\|M_\delta(g)\|_{L^{p'}_\delta}.$$

Now, since we are working on the real line, we have that $\|M_\delta(g)\|_{L^{p'}_\delta}\leq C_p\|g\|_{L^{p'}_\delta}$ holds for every weight δ, with $(C_p)^{p'}=2^{p'+1}p$ and therefore

$$\int\left(\sum\lambda_k\chi_{I_k}(y)\right)g(y)\delta(y)dy\leq C_p\eta^{-1}\left\|\sum\lambda_k\chi_{E_k}\right\|_{L^p_\delta}.$$

From this inequality we obtain immediately the conclusion for $p\geq 1$.

Now, let us prove the case $0<p<1$. By denoting $\Psi=\sum\lambda_k\chi_{E_k}$ and $\Phi=\sum\lambda_k\chi_{I_k}$ and, for a fixed $t>0$, we define

$$\mathcal{E}=\{x\in\mathbb{R}:\Psi(x)>t\}\quad\text{and}\quad\mathcal{O}=\left\{x\in\mathbb{R}:M_\delta\chi_{\mathcal{E}}(x)>\frac{\eta}{2}\right\}.$$

Since we are working on the real line, M_δ is weak type $(1,1)$ with constant 2 with respect to the weight δ. So, we have

$$\delta(\mathcal{O})\leq\frac{4}{\eta}\delta(\mathcal{E}).\tag{4}$$

If $\mathcal{O}^c \cap I_k \neq \varnothing$ then $\delta(\mathcal{E} \cap E_k) \leq \delta(\mathcal{E} \cap I_k) \leq \frac{\eta}{2}\delta(I_k) \leq \frac{1}{2}\delta(E_k)$ and consequently,

$$\delta(E_k) \leq \frac{1}{2}\delta(E_k) + \delta(\mathcal{E}^c \cap E_k).$$

Therefore

$$\eta\delta(\mathcal{E}^c \cap I_k) \leq \delta(E_k) \leq 2\delta(\mathcal{E}^c \cap E_k) \quad \text{if } \mathcal{O}^c \cap I_k \neq \varnothing. \tag{5}$$

Taking $r > 1$ and using that $\mathcal{O}^c \subseteq \mathcal{E}^c$,

$$\int_{\mathcal{O}^c} \Phi(x)^r \delta(x)\,dx \leq \int_{\mathcal{E}^c} \left(\sum_{\mathcal{O}^c \cap I_k \neq \varnothing} \lambda_k \chi_{I_k}(x) \right)^r \delta(x)\,dx.$$

Since (5) holds, we can apply, with the weight $\delta(x)\chi_{\mathcal{E}^c}(x)$, the case $r > 1$ we have just proved, to estimate the last term by

$$C_r \eta^{-1} \int_{\mathcal{E}^c} \left(\sum_{\mathcal{O}^c \cap I_k \neq \varnothing} \lambda_k \chi_{E_k}(x) \right)^r \delta(x)\,dx.$$

So, we have

$$\int_{\mathcal{O}^c} \Phi(x)^r \delta(x)\,dx \leq C_r \eta^{-1} \int_{\mathcal{E}^c} \Psi(x)^r \delta(x)\,dx. \tag{6}$$

From $\delta(\{x : \Phi(x) > t\}) \leq \delta(\mathcal{O}) + \delta(\mathcal{O}^c \cap \{x : \Phi(x) > t\})$ and by (4) we have

$$\delta(\{x : \Phi(x) > t\}) \leq \frac{4}{\eta}\delta(\mathcal{E}) + \frac{1}{t^r} \int_{\mathcal{O}^c} \Phi(x)^r \delta(x)\,dx.$$

By (6) we obtain

$$\delta(\{x : \Phi(x) > t\}) \leq \frac{4}{\eta}\delta(\mathcal{E}) + \frac{C_r}{\eta t^r} \int_{\mathcal{E}^c} \Psi(x)^r \delta(x)\,dx.$$

From the estimation above, we get

$$\left\| \sum \lambda_k \chi_{I_k} \right\|_{L^p_\delta}^p = \int_0^{+\infty} p t^{p-1} \delta(\{x : \Phi(x) > t\})\,dt$$
$$\leq \frac{4}{\eta} \int_0^{+\infty} p t^{p-1} \delta(\mathcal{E})\,dt + \frac{C_r}{\eta} \int_0^{+\infty} p t^{p-1-r} \int_{\mathcal{E}^c} \Psi(x)^r \delta(x)\,dx\,dt.$$

So, the lemma follows since for $0 < p < 1$ the last term equals

$$\left(\frac{4}{\eta} + \frac{C_r}{\eta} \frac{p}{r-p} \right) \left\| \sum \lambda_k \chi_{E_k} \right\|_{L^p_\delta}^p.$$

■

The next lemma is contained in Theorem 1 of [2].

Lemma 6 *Let $\delta \in A_\infty^+$.*
1) There exists $\beta > 0$ such that the following implication holds: given $\lambda > 0$ and an interval (a,b) such that $\lambda \leq \frac{\delta(a,x)}{x-a}$ for all $x \in (a,b)$, then:

$$|\{x \in (a,b) : \delta(x) > \beta\lambda\}| > \frac{1}{2}(b-a).$$

2) There exists $\gamma > 0$ such that the following implication holds: given $\lambda > 0$ and an interval (a,b) such that $\lambda \geq \frac{\delta(x,b)}{b-x}$ for all $x \in (a,b)$, then:

$$\delta(\{x \in (a,b) : \delta(x) < \gamma\lambda\}) > \frac{1}{2}\delta(a,b).$$

4 An appropriate atomic decomposition

In this section we give an atomic decomposition of a distribution $f \in H_+^p(\omega)$ with additional properties that we shall need.

We shall say that an interval J 'follows' the interval I if $I = [c,d]$ and $J = [d,e]$.

Our goal is to prove that given $f \in H_+^p(\omega)$ there is an atomic decomposition as stated in Theorem 1 such that for every atom a_k supported in an interval I_k there is another atom a_j supported in an interval I_j following I_k.

First we shall need a couple of lemmas in order to 'break up' an atom.

Lemma 7 *Let $r > 0$. There exists a sequence $\{\eta_j\}_{j=-\infty}^{+\infty}$ of C_0^∞ functions such that*

1) $0 \leq \eta_j \leq 1$ and $\sum_j \eta_j(x) = \chi_{(-\infty,r)}(x)$.
2) $\text{supp}(\eta_j) \subset I_j = \left[r - 2^{-j}r, r - 2^{-j-2}r\right]$.
3) If we denote $r_j = \frac{r}{2^j}$ and $x \in I_j$ then $\frac{1}{4}r_j \leq r - x \leq r_j$.
4) Each x belongs to at most three intervals I_j.
5) For every non negative integer i there exists a positive constant c_i such that $\left|D^i\eta_j(x)\right| \leq c_i r_j^{-i}$.

Proof. Let

$$h(y) = \int_y^{y/2} \rho(t)dt,$$

where ρ is a non negative $C_0^\infty(\mathbb{R})$ function with support contained in $[-2,-1]$ and $\int \rho(t)dt = 1$.We define

$$\eta_j(x) = h\left(\frac{x-r}{2^{-j-2}r}\right).$$

It is not hard to see that $\{\eta_j\}$ satisfy the five conditions. For details see [3]. ■

Lemma 8 *Let $a(y)$ be an (∞, N)-atom with support contained in an interval I. There exists a sequence $\{a_j(x)\}$ of (∞, N)-atoms with associated intervals $\{I_j\}$ such that $a(x) = c\sum_j a_j(x)$ almost everywhere, with c a positive constant independent of $a(x)$. In addition $I = \cup_j I_j$ and no point $x \in I$ belongs to more than three intervals I_j.*

Proof. Without lost of generality, we can assume that $I = [0, r]$. Since $a(y)$ is bounded with compact support,

$$A(x) = \frac{1}{N!}\int_x^\infty (y-x)^N a(y)dy,$$

it is well defined.

The vanishing moments condition of $a(x)$ implies that $\mathrm{supp}(A) \subset [0, r]$. Moreover, it is not hard to see that $D^{N+1}A(x) = (-1)^{N+1} a(x)$ for almost every x.

Let $\{\eta_j\}_{j=-\infty}^{\infty}$ be the functions of Lemma 7 associated to the interval $(-\infty, r)$. Then, by condition 1) of Lemma 7, we have

$$A(x) = \sum_{j=-1}^{\infty} A(x)\eta_j(x). \tag{7}$$

If we denote $A_j(x) = A(x)\eta_j(x)$ and $b_j(x) = (-1)^{N+1}D^{N+1}A_j(x)$, we have that $\mathrm{supp}(b_j) \subset [0, r] \cap [r - 2^{-j}r, r - 2^{-j-2}r] = I_j$, and, since $\mathrm{supp}(A_j)$ is bounded, it can be shown by integration by parts that $b_j(x)$ has N vanishing moments.

We claim that $\|b_j\|_\infty \leq c$, where c only depends on N. By Leibniz's formula we have

$$D^{N+1}A_j(x) = \sum_{k=0}^{N+1} c_{k,N} D^k A(x) D^{N+1-k}\eta_j(x). \tag{8}$$

For $x \in \mathrm{supp}(\eta_j)$ and $k \leq N$, since $\|a\|_\infty \leq 1$, we obtain

$$\left|D^k A(x)\right| \leq c\int_x^r (t-x)^{N-k} |a(t)|\, dt \leq c\,(r-x)^{N+1-k}.$$

Thus, from (8) and conditions 5) and 3) of Lemma 7, we get

$$\begin{aligned} |b_j(x)| = \left|D^{N+1}A_j(x)\right| &\leq c_N \sum_{k=0}^{N+1} (r-x)^{N+1-k} \left|D^{N+1-k}\eta_j(x)\right| \\ &\leq c_N \sum_{k=0}^{N+1} \frac{(r-x)^{N+1-k}}{r_j^{N+1-k}} \leq c_N\,(N+2) = c. \end{aligned} \tag{9}$$

Furthermore, as a consequence of (7), we have,

$$a(x) = c\sum_{j=-1}^{\infty} a_j(x),$$

a.e x, where $a_j(x) = \frac{b_j(x)}{c}$ with c as in (9) are (∞, N) atoms. ■

Remark 9 *We observe that I_{j+2} follows I_j and that $|I_{j+2}| \le |I_j| \le 4\,|I_{j+2}|$.*

Taking into account this remark and as a consequence of Theorem 1 and the previous lemma we have the following result:

Theorem 10 *Let $\omega \in A_s^+$ and $0 < p < \infty$. Then there is a integer $N(p,\omega)$ with the following property: given any $f \in H_+^p(\omega)$ and $N \ge N(p,\omega)$, we can find a sequence $\{\lambda_k\}$ of positive coefficients and a sequence $\{a_{k,j}(x)\}$ of (∞,N)-atoms with support contained in intervals $\{I_{k,j}\}$ respectively such that the sum $\sum_{k,j} \lambda_k a_{k,j}$ converges unconditionally to f both in the sense of distributions and in the $H_+^p(\omega)$-norm. Moreover,*

$$\|f\|_{H_+^p(\omega)} \sim \left\| \sum_{k,j} \lambda_k \chi_{I_{k,j}} \right\|_{L_\omega^p}$$

and,for every j, k, $I_{k,j+2}$ follows $I_{k,j}$, and $|I_{k,j+2}| \le |I_{k,j}| \le 4\,|I_{k,j+2}|$.

5 Proof of Theorem 3

The first part of the theorem can be obtained as in the proof of Theorem 3 in Chapter XII of [6] using the maximal function $M_1^+(f,\phi,x)$ defined on [4]. For the second part it is enough to define $f(z)$ and to prove inequality (3).
Let $f \in H_+^p(\mu)$ then there exists an atomic decomposition $f = \sum_{k,j} \lambda_k a_{k,j}$ as the one given in Theorem 10. With the notation introduced in section 2 and for $z \in \Omega$, we define

$$f(z) = \sum_{k,j} \lambda_{k,j}(z) a_{k,j},$$

where

$$\lambda_{k,j}(z) = \lambda_k^{p/p(z)} \left(\frac{\omega(I_{k,j+2})}{\nu(I_{k,j+2})} \right)^{\frac{(z-s)p}{p_0 p_1}} .$$

By Theorem 1 there exists a constant C such that

$$\|f(u+it)\|_{H_+^{p(u)}(\mu(u))} \le C \left\| \sum |\lambda_{k,j}(u+it)| \chi_{I_{k,j}} \right\|_{L_{\mu(u)}^{p(u)}},$$

as long as the second term is finite. Then we shall prove that

$$\left\| \sum |\lambda_{k,j}(u+it)| \chi_{I_{k,j}} \right\|_{L_{\mu(u)}^{p(u)}}^{p(u)} \le C\, \|f\|_{H_+^p(\mu)}^p < \infty.$$

First we claim that there exist $\beta_{k,j} \in I_{k,j+2}$ and a constant c such that, for every $x \in I_{k,j} \cup I_{k,j+2}$,

$$\frac{\mu(u)\,(x,\beta_{k,j})}{\beta_{k,j} - x} \le 4c \frac{\mu(u)(I_{k,j+2})}{|I_{k,j+2}|}. \tag{10}$$

In fact, since M^- is weak type $(1,1)$ with constant 2 with respect to the Lebesgue measure, we have, for every k, j, that

$$\left|\left\{x \in I_{k,j+2} : M^-(\mu(u)\chi_{I_{k,j}\cup I_{k,j+2}})(x) > 4\frac{\mu(u)(I_{k,j}\cup I_{k,j+2})}{|I_{k,j+2}|}\right\}\right| \leq \frac{|I_{k,j+2}|}{2}.$$

So, there exists $\beta_{k,j} \in I_{k,j+2}$ such that

$$M^-(\mu(u)\chi_{I_{k,j}\cup I_{k,j+2}})(\beta_{k,j}) \leq 4c\frac{\mu(u)(I_{k,j+2})}{|I_{k,j+2}|},$$

where c is the left doubling constant of $\mu(u)$. This implies (10).
We denote by $\alpha_{k,j}$ the left end point of $I_{k,j}$ and

$$E_{k,j} = \left\{x \in (\alpha_{k,j}, \beta_{k,j}) : \mu(u)(x) < \gamma 4c\frac{\mu(u)(I_{k,j+2})}{|I_{k,j+2}|}\right\},$$

where γ is the constant given in part 2 of Lemma 6. By (10) we can apply part 2 of Lemma 6 with $\lambda = 4c\frac{\mu(u)(I_{k,j+2})}{|I_{k,j+2}|}$ to obtain

$$\mu(u)(E_{k,j}) \geq \frac{\mu(u)(\alpha_{k,j}, \beta_{k,j})}{2}. \tag{11}$$

Since $I_{k,j} \subset (\alpha_{k,j}, \beta_{k,j})$, we have

$$\left\|\sum |\lambda_{k,j}(u+it)|\chi_{I_{k,j}}\right\|_{L^{p(u)}_{\mu(u)}} \leq \left\|\sum |\lambda_{k,j}(u+it)|\chi_{(\alpha_{k,j},\beta_{k,j})}\right\|_{L^{p(u)}_{\mu(u)}}.$$

From (11) we can apply Lemma 5, and estemate the last term by

$$C\left\|\sum |\lambda_{k,j}(u+it)|\chi_{E_{k,j}}\right\|_{L^{p(u)}_{\mu(u)}} = C\left\|\sum |\lambda_{k,j}(u+it)|\mu(u)\,(\cdot)^{\frac{1}{p(u)}}\,\chi_{E_{k,j}}\right\|_{L^{p(u)}}.$$

By the definition of $\lambda_{k,j}(u+it)$, taking into account that if $x \in E_{k,j}$ then

$$\mu(u)(x) < \gamma 4c\frac{\mu(u)(I_{k,j+2})}{|I_{k,j+2}|}$$

and from (2), the last term is bounded by

$$C\left\|\sum \lambda_k^{\frac{p}{p(u)}}\left[\frac{\omega(I_{k,j+2})}{\nu(I_{k,j+2})}\right]^{\frac{(u-s)p}{p_0p_1}}\left[\frac{1}{|I_{k,j+2}|}\int_{I_{k,j+2}}\omega(x)^{1-\mu(u)}\nu(x)^{\mu(u)}dx\right]^{\frac{1}{p(u)}}\chi_{E_{k,j}}\right\|_{L^{p(u)}}.$$

Since $\frac{(u-s)p}{p_0 p_1} = \frac{\mu(u)-\mu}{p(u)}$ and using Hölder's inequality the last expression is bounded by

$$C\left\|\sum \lambda_k^{\frac{p}{p(u)}} \left(\frac{\omega(I_{k,j+2})}{\nu(I_{k,j+2})}\right)^{\frac{\mu(u)-\mu}{p(u)}} \left[\frac{\omega(I_{k,j+2})^{1-\mu(u)}\nu(I_{k,j+2})^{\mu(u)}}{|I_{k,j+2}|}\right]^{\frac{1}{p(u)}} \chi_{E_{k,j}}\right\|_{L^{p(u)}}$$

$$\leq C\left\|\sum \lambda_k^{\frac{p}{p(u)}} \left(\frac{\omega(I_{k,j+2})^{1-\mu}\nu(I_{k,j+2})^{\mu}}{|I_{k,j+2}|}\right)^{\frac{1}{p(u)}} \chi_{I_{k,j}\cup I_{k,j+2}\cup I_{k,j+4}\cup I_{k,j+6}}\right\|_{L^{p(u)}}.$$

Therefore, by Lemma 4 and Lemma 5, we get

$$\begin{aligned}\|f(u+it)\|_{H_+^{p(u)}(\mu(u))} &\leq C\left\|\sum \lambda_k^{\frac{p}{p(u)}} \left(\frac{\mu(I_{k,j+4})}{|I_{k,j+4}|}\right)^{\frac{1}{p(u)}} \chi_{I_{k,j+6}}\right\|_{L^{p(u)}} \\ &\leq C\left\|\sum \lambda_k^{\frac{p}{p(u)}} \left(\frac{\mu(I_{k,j+2})}{|I_{k,j+2}|}\right)^{\frac{1}{p(u)}} \chi_{I_{k,j+4}}\right\|_{L^{p(u)}}.\end{aligned}$$

Now we shall consider $M_\mu^+ g\,(x) = \sup_{h>0} \frac{1}{\mu(x,x+h)} \int_x^{x+h} g\,(t)\,\mu(t)dt$. Since M_μ^+ is weak $(1,1)$ with constant 2 with respect to the weight μ we have

$$\mu\left(\left\{x \in I_{k,j+2} : M_\mu^+(\mu^{-1}\chi_{I_{k,j+2}\cup I_{k,j+4}})(x) > 4\frac{|I_{k,j+2}\cup I_{k,j+4}|}{\mu(I_{k,j+2})}\right\}\right) \leq \frac{\mu\,(I_{k,j+2})}{2},$$

so we can choose $c_{k,j} \in I_{k,j+2}$ such that

$$M_\mu^+(\mu^{-1}\chi_{I_{k,j+2}\cup I_{k,j+4}})(c_{k,j}) \leq 8\frac{|I_{k,j+2}|}{\mu(I_{k,j+2})}. \tag{12}$$

Since, for every $x \in I_{k,j+2}\cup I_{k,j+4}$, $x > c_{k,j}$, $\frac{x-c_{k,j}}{\mu(c_{k,j},x)} \leq M_\mu^+(\mu^{-1}\chi_{I_{k,j+2}\cup I_{k,j+4}})(c_{k,j})$, we have from (12) that

$$\frac{\mu(I_{k,j+2})}{8\,|I_{k,j+2}|} \leq \frac{\mu(c_{k,j},x)}{x-c_{k,j}}.$$

Then if we denote by $d_{k,j}$ the right end point of $I_{k,j+4}$ and

$$F_{k,j} = \left\{x \in (c_{k,j}, d_{k,j}) : \mu(x) > \frac{\beta\ \mu(I_{k,j+2})}{4|I_{k,j+2}|}\right\},$$

we can apply part 1 of Lemma 6 to obtain

$$|F_{k,j}| \geq \frac{(d_{k,j}-c_{k,j})}{2}.$$

So, by Lemma 5 and definition of $F_{k,j}$,

$$\left\|\sum \lambda_k^{\frac{p}{p(u)}} \left(\frac{\mu(I_{k,j+2})}{|I_{k,j+2}|}\right)^{\frac{1}{p(u)}} \chi_{I_{k,j+4}}\right\|_{L^{p(u)}} \leq C\left\|\sum \lambda_k^{\frac{p}{p(u)}} \left(\frac{\mu(I_{k,j+2})}{|I_{k,j+2}|}\right)^{\frac{1}{p(u)}} \chi_{F_{k,j}}\right\|_{L^{p(u)}}$$

$$\leq C\left\|\sum \lambda_k^{\frac{p}{p(u)}} \mu\,(\cdot)^{\frac{1}{p(u)}} \chi_{F_{k,j}}\right\|_{L^{p(u)}} \leq C\left\|\sum \lambda_k^{\frac{p}{p(u)}} \chi_{I_{k,j+4}}\right\|_{L_\mu^{p(u)}}.$$

Therefore,

$$\|f(u+it)\|_{H_+^{p(u)}(\mu(u))} \leq C \left\| \sum \lambda_k^{\frac{p}{p(u)}} \chi_{I_{k,j}} \right\|_{L_\mu^{p(u)}}.$$

By a density argument such as the one given in page 187 of [6], we can assume that $f \in L^1_{loc}(\mathbb{R})$. Thus, using (1) in the last inequality, we have

$$\|f(u+it)\|^{p(u)}_{H_+^{p(u)}(\mu(u))} \leq C \left\| f_N^{*\,\frac{p}{p(u)}} \right\|^{p(u)}_{L_\mu^{p(u)}} = C\,\|f\|^p_{H_+^p(\mu)},$$

as we wanted to prove.

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