ONE-SIDED SINGULAR INTEGRAL OPERATORS ON CALDERÓN-HARDY SPACES

S. OMBROSI AND C. SEGOVIA

Abstract. In [5] we have defined and studied the $\mathcal{H}^{p,+}_{q,\alpha}(w)$ spaces for weights w belonging to the class A^+_s defined by E. Sawyer, and where the parameter α is a positive real number. When α is a natural number, these spaces can be identified with the one-sided Hardy space $H^p_+(w)$ defined in [7]. This identification could be used to define a continuous extension of a one-sided regular Calderón-Zygmund operator from $\mathcal{H}^{p,+}_{q,\alpha}(w)$ into $\mathcal{H}^{p,+}_{q,\alpha}(w)$, when the parameter α is a natural number. In this paper, we give a direct definition of a one-sided regular Calderón-Zygmund operator on $\Lambda_\alpha \cap \mathcal{H}^{p,+}_{q,\alpha}(w)$, which is valid for any real number $\alpha > 0$, and we prove that these operators can be extended to bounded operators from $\mathcal{H}^{p,+}_{q,\alpha}(w)$ into $\mathcal{H}^{p,+}_{q,\alpha}(w)$.

1. Notation, definitions and some previous results

Let $f(x)$ be a Lebesgue measurable function defined on $\mathbb{R}$. The one-sided Hardy-Littlewood maximal functions $M^+f(x)$ and $M^-f(x)$ are defined as

$$M^+f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f(t)|\, dt \qquad \text{and} \qquad M^-f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f(t)|\, dt.$$

As usual, a weight $w(x)$ is a measurable and non-negative function. If $E \subset \mathbb{R}$ is a Lebesgue measurable set, we denote its w-measure by $w(E) = \int_E w(t)dt$. A function $f(x)$ belongs to $L^s(w)$, $0 < s \le \infty$, if $\|f\|_{L^s(w)} = \left(\int_{-\infty}^{\infty} f(x)^s w(x) dx\right)^{1/s}$ is finite.

A weight $w(x)$ belongs to the class A^+_s, $1 \le s < \infty$, defined by E. Sawyer in [7], if there exists a constant c such that

$$\sup_{h>0} \left(\frac{1}{h}\int_{x-h}^{x} w(t)dt\right)\left(\frac{1}{h}\int_x^{x+h} w(t)^{-\frac{1}{s-1}} dt\right)^{s-1} \le c,$$

for all real number x. We observe that $w(x)$ belongs to the class A^+_1 if and only if $M^-w(x) \le cw(x)$ for all real number x. It is well known that if $w(x) \in A^+_s$ $(1 < s < \infty)$, then there exists a constant c_w such that the inequality

$$\left\|M^+f\right\|_{L^s(w)} \le c_w \|f\|_{L^s(w)} \tag{1}$$

holds for every $f \in L^s(w)$ (e.g., see [7] or [4]).

Given $w(x) \in A^+_s$, $1 \le s < \infty$, we can define a number $x_{-\infty}$, $-\infty \le x_{-\infty} \le \infty$, such that for almost every x, $w(x) = 0$ in $(-\infty, x_{-\infty})$ and $0 < w(x)$ in $(x_{-\infty}, +\infty)$.

Let us fix $w \in A^+_s$ and let $x_{-\infty}$ be as before. Let $L^q_{loc}(x_{-\infty}, \infty)$, $1 < q < \infty$, be the space of the real-valued functions $f(x)$ on $\mathbb{R}$ that belong locally to L^q for compact subsets of $(x_{-\infty}, \infty))$. We endow $L^q_{loc}(x_{-\infty}, \infty)$ which the topology generated for the seminorms

$$|f|_{q,I} = \left(|I|^{-1} \int_I |f(y)|^q\, dy\right)^{1/q},$$

Key words and phrases. **one-sided weights, one-sided regular Calderón-Zygmund kernels, Calderón-Hardy spaces.**

2000 *Mathematics Subject Classification* Primary: 42B20; Secondary: 42B35.

This research has been partially supported by UBACYT 2000-2002, CONICET and FOMEC.

where $I=[a,b]$ is an interval contained in $(x_{+\infty},\infty)$ and $|I|=b-a$.

For $f(x)$ in $L^q_{loc}(x_{-\infty},\infty)$, we define a maximal function $n^+_{q,\alpha}(f;x)$ as

$$n^+_{q,\alpha}(f;x)=\sup_{\rho>0}\rho^{-\alpha}\,|f|_{q,[x,x+\rho]}\,,$$

where α is a positive real number.

Let N a non negative integer and $\mathcal{P}_N$ the subspace of $L^q_{loc}(x_{-\infty},\infty)$ formed by all the polynomials of degree at most N. We denote by E^q_N the quotient space of $L^q_{loc}(x_{-\infty},\infty)$ by $\mathcal{P}_N$. If $F\in E^q_N$, we define the seminorm $\|F\|_{q,I}=\inf\left\{|f|_{q,I}:f\in F\right\}$. The family of all these seminorms induces on E^q_N the quotient topology.

Given a real number $\alpha>0$, we can write $\alpha=N+\beta$, where N is a non negative integer and $0<\beta\le 1$. This decomposition is unique.

For F in E^q_N, we define a maximal function $N^+_{q,\alpha}(F;x)$ as

$$N^+_{q,\alpha}(F;x)=\inf\left\{n^+_{q,a}(f;x):f\in F\right\}.$$

We say that an element F in E^q_N belongs to the Calderón-Hardy space $\mathcal{H}^{p,+}_{q,\alpha}(w)$, $0<p\le 1$, if the maximal function $N^+_{q,\alpha}(F;x)\in L^p(w)$. The "norm" of F in $\mathcal{H}^{p,+}_{q,\alpha}(w)$ is defined as $\|F\|_{\mathcal{H}^{p,+}_{q,\alpha}(w)}=\left\|N^+_{q,\alpha}(F;x)\right\|_{L^p(w)}$. These spaces have been defined in [5] and, in the case that $w=1$, these spaces have been studied in [3].

We say that a class $A\in E^q_N$ is a p-atom in $\mathcal{H}^{p,+}_{q,\alpha}(w)$ if there exist a representative $a(y)$ of A and a bounded interval I such that

i) $\operatorname{supp}(a)\subset I\subset(x_{-\infty},\infty)$, $w(I)<\infty$

ii) $N^+_{q,\alpha}(A,x)\le w(I)^{-1/p}$ for all $x\in(x_{-\infty},\infty)$.

In [5] it was proved the following result:

Theorem 1.1 (Descomposition into atoms). *Let $w\in A^+_s$ and $0<p\le 1$, such that $(\alpha+1/q)\,p\ge s>1$ or $(\alpha+1/q)\,p>1$ if $s=1$. Then, if $F\in\mathcal{H}^{p,+}_{q,\alpha}(w)$ there exists a sequence $\{\lambda_i\}$ of the number real and a sequence $\{A_i\}$ of p-atoms in $\mathcal{H}^{p,+}_{q,\alpha}(w)$ such that $F=\sum\lambda_iA_i$ en $E^q_N(x_{-\infty},\infty)$. Moreover the series $\sum\lambda_iA_i$ converges in $\mathcal{H}^{p,+}_{q,\alpha}(w)$ and there exist two constants c_1 and c_2 not depending of F, such that $c_1\,\|F\|^p_{\mathcal{H}^{p,+}_{q,\alpha}(w)}\le\sum|\lambda_i|^p\le c_2\,\|F\|^p_{\mathcal{H}^{p,+}_{q,\alpha}(w)}$.*

As before, let $\alpha=N+\beta$, where $0<\beta\le 1$. We denote by $\Lambda_\alpha\,(x_{-\infty},\infty)$, the space consisting of those classes F in E^q_N such that if $f\in F$ then $f\in C^N\,(x_{-\infty},\infty)$, and there exists a constant C such that the derivative D^Nf satisfies the Lipschitz condition

$$\left|D^Nf(x)-D^Nf(y)\right|\le C\ \ |y-x|^\beta\ \text{ for every } x,y \text{ in } (x_{-\infty},\infty)\,.$$

To simplify the notation, we write Λ_α instead $\Lambda_\alpha(x_{-\infty},\infty)$. In the following lemma we state some results on the maximal function $N^+_{q,\alpha}(F,x)$ and the spaces $\mathcal{H}^{p,+}_{q,\alpha}(w)$ that we will need in this paper.

Lemma 1.2. *Let $F\in E^q_N$.*

(i) If $N^+_{q,\alpha}\,(F,x_0)$ is finite for some x_0 there exists a unique representative f of F such that $N^+_{q,\alpha}(F,x_0)=n^+_{q,\alpha}(f,x_0)$.

(ii) F belongs to Λ_α if and only if there exists a constant finite C such that $N^+_{q,\alpha}(F,x)\le C$ for all $x\in(x_{-\infty},\infty)$.

(*iii*) *If* $F \in \mathcal{H}^{p,+}_{q,\alpha}(w)$ *and* $t > 0$, *we can decompose* F *as* $F = G_t + \Theta_t$, *where* $N^+_{q,\alpha}(G_t, x) \leq C\ t$ *for all* $x \in (x_{-\infty}, \infty)$ *and*

$$\int_{x_{-\infty}}^{\infty} N^+_{q,\alpha}(\Theta_t, x)^p w(x)dx \leq C \int_{\left\{x\in(x_{-\infty},\infty):N^+_{q,\alpha}(F,x)>t\right\}} N^+_{q,\alpha}(F, x)^p w(x)dx.$$

Proof. Part (*i*) is Lemma 2.2 in [5], part (*ii*) is Lemma 3.10 in [5] and part (*iii*) is Lemma 4.3 in [5]. □

Corollary 1.3. *The set* $\mathcal{H}^{p,+}_{q,\alpha}(w) \cap \Lambda_\alpha$ *is dense in* $\mathcal{H}^{p,+}_{q,\alpha}(w)$.

We say that a function k in $L^1_{loc}(\mathbb{R} - \{0\})$ is a regular Calderón-Zygmund kernel, if there exists a finite constant C such that the following properties are satisfied:

(a) $\left|\int_{\varepsilon<|x|<M} k(x)dx\right| \leq C$ holds for all ε and M, $0 < \varepsilon < M$, and there exists $\lim_{\varepsilon\to 0^+} \int_{\varepsilon<|x|<1} k(x)dx$.

(b) $|k(x)| \leq \frac{C}{|x|}$, for all $x \neq 0$.

(c) $|k(x - y) - k(x)| \leq C\,|y|\,|x|^{-2}$ for all x and y with $|x| > 2\,|y| > 0$.

We observe that (b) implies that for $r > 0$,

$$\int_{r\leq|y|\leq 2r} |k(y)|\, dy \leq C \int_{r\leq|y|\leq 2r} |y|^{-1}\, dy \leq C'. \tag{2}$$

A regular Calderón-Zygmund kernel with support in $(-\infty, 0)$ will be called a one-sided regular Calderón-Zygmund kernel. In [1] H. Aimar, L. Forzani and F. Martín-Reyes proved that the class of these kernels is not empty, in fact, the kernel

$$k(x) = \frac{\sin(\log|x|)}{|x|\log|x|}\chi_{(-\infty,0)}(x), \tag{3}$$

satisfies the conditions (a), (b) and (c).

We denote

$$Kf(x) = v.p. \int k(x - y)f(y)dy = \lim_{\varepsilon\to 0^+} \int_{|x-y|>\varepsilon} k(x - y)f(y)dy,$$

the singular integral operator associated with $k(y)$, and by $K^*f(x)$ the maximal singular integral operator given by

$$K^*f(x) = \sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} k(x - y)f(y)dy \right|. \tag{4}$$

The following result can be found in [1].

Theorem 1.4 ([1])**.** *Let* $w \in A^+_s$, $1 < s < \infty$, *and let* k *be a one-sided regular Calderón-Zygmund kernel. Then, there exists a finite constant* C *such that*

$$\int |K^*f(x)|^s\, w(x)dx \leq C \int |f(x)|^s\, w(x)dx$$

holds for all $f \in L^s(w)$.

Let n be a non negative integer, we will say that $k(x)$ is a regular kernel of order n, if $k \in C^n$ away the origin, and

$$\left|D^i k(x)\right| \leq \frac{C_i}{|x|^{i+1}}, \text{ for every } i = 1, 2, ..., n \text{ and every } x \neq 0 \tag{5}$$

Lemma 1.5. *The kernel* $k\,(x)$ *defined in* (3) *is regular of order* n, *for every* $n \geq 0$.

Proof. We denote $g(t) = \frac{\sin t}{t}$ and $f(t) = \log|t|$. For $x < 0$, we get

$$k(x) = -(g \circ f(x))\, Df(x).$$

Now, since $Df(x) = \frac{1}{x}$, we have that

$$D^i f(x) = (-1)^{i-1}(i-1)!\, [Df(x)]^i,$$

for every natural number i. Arguing by induction it is easy to see that if n is natural number, then $D^n k(x)$ is given by a sum of $n+1$ terms of the way

$$C_{h,n} D^h g \circ f(x)\, [Df(x)]^{n+1},$$

where $C_{h,n}$ is a constant and $0 \le h \le n$. Then, since $D^h g(t) \in L^\infty$ for every non negative integer h, the lemma follows. □

2. Definition of one-sided regular Calderón-Zygmund operators on the classes $\mathcal{H}^{p,+}_{q,\alpha}(w) \cap \Lambda_\alpha$

We will assume in the sequel that $w \in A^+_s$, where $(\alpha + 1/q)p \ge s > 1$ or $(\alpha + 1/q)p > 1$ if $s = 1$; and without loss of generality, we will assume that the number $x_{-\infty}$ associated to the weight w is less than zero.

Lemma 2.1. *Let $\alpha = N+1$ and let $F \in \mathcal{H}^{p,+}_{q,\alpha}(w) \cap \Lambda_\alpha$. If $f \in F$ then*

$$\left|D^{N+1} f(x)\right| \le N^+_{q,\alpha}(F;x) \text{ for every } x \in (x_{-\infty}, \infty) \tag{6}$$

The proof of this lemma is similar to the proof of Theorem 4 in [2], and it will not be given here.

Lemma 2.2. *Let F in Λ_α and $x_1 \in (x_{-\infty}, \infty)$. If $f(y)$ is the representative of F such that $N^+_{q,\alpha}(F, x_1) = n^+_{q,\alpha}(f, x_1)$, then*

$$\left|D^i f(y)\right| \le C \left\|N^+_{q,\alpha}(F;.)\right\|_\infty |y - x_1|^{\alpha - i} \text{ holds, for } i = 0, 1, ..., N \text{ and } y \in (x_{-\infty}, \infty). \tag{7}$$

Proof. The proof of this result is a corollary of the proof of Lemma 4.2 in [5]. In fact, with the notation of that lemma, if we consider $t = \left\|N^+_{q,\alpha}(F;.)\right\|_\infty$, then F coincides with the class G that appear there. Then (7) follows from estimate (24) of Lemma 4.2 in [5]. □

Let us fix a function $\phi \in C_0^\infty$, $0 \le \phi(y) \le 1$, $\mathrm{supp}(\phi) \subset [-2, 2]$ and such that $\phi(y) \equiv 1$ in $[-1, 1]$. Let $r > 0$, and $x_1 \in \mathbb{R}$. We denote

$$\phi_{x_1,r}(y) = \phi\left(\frac{y - x_1}{r}\right). \tag{8}$$

Then, the support of $\phi_{x_1,r}(y)$ is contained in $[x_1 - 2r, x_1 + 2r]$ and $\phi(y) \equiv 1$ in $[x_1 - r, x_1 + r]$. Moreover, we have that

$$\left|D^i(\phi_{x_1,r})(y)\right| \le C_i r^{-i}, \tag{9}$$

for every non negative integer i. If $x_1 = 0$, we denote $\phi_{0,r}(y)$ by $\phi_r(y)$.

Lemma 2.3. *Let $\alpha = N+1$, and $F \in \mathcal{H}^{p,+}_{q,\alpha}(w) \cap \Lambda_\alpha$. Let $f(y)$ be the representative of F such that $n^+_{q,\alpha}(f, 0) = N^+_{q,\alpha}(F, 0)$. If $k(y)$ is a one-sided regular Calderón-Zygmund kernel, then*

$$\lim_{j \to +\infty} \left| \int k(-y) D^i f(y) D^{N+1-i} \phi_j(y) dy \right| = 0, \text{ for } i = 0, 1, ..., N,$$

and $\phi_j(y) = \phi(\frac{y}{j})$, where ϕ is the function that was fixed before.

Proof. By Lemma 2.2, it follows that $D^h f(0) = 0$, for $h = 0, 1, ..., N$. Then, by the Taylor's formula and Lemma 2.1, we obtain

$$\begin{aligned} \left|D^i f(y)\right| &= \left|D^i f(y) - \sum_{h=0}^{N-i} D^{i+h} f(0) \frac{y^h}{h!}\right| \le C \int_0^1 \left|D^{N+1} f(ty)\right| (1-t)^{N+1-i} dt \ |y|^{N+1-i} \\ &\le C \int_0^1 N_{q,\alpha}^+(F; ty) dt \ |y|^{N+1-i}. \end{aligned}$$

From the last estimate, since $\operatorname{supp}(k) \subset (-\infty, 0)$ and $\operatorname{supp}\left(D^{N+1-i}\phi_j\right) \subset \{j \le |y| \le 2j\}$, we have that

$$\left|\int k(-y) D^i f(y) D^{N+1-i} \phi_j(y) dy\right| \tag{10}$$

$$\le C \int_j^{2j} \left|D^{N+1-i}\phi_j(y)\right| |k(-y)| \, |y|^{N+1-i} \int_0^1 N_{q,\alpha}^+(F; ty) dt \ dy.$$

By (9), we have that $\left|D^{N+1-i}\phi_j(y)\right| |y|^{N+1-i} \le C$, if $|y| \le 2j$. From this fact and by (10), we obtain

$$\left|\int k(-y) D^i f(y) D^{N+1-i} \phi_j(y) dy\right| \le C \int_0^1 \int_j^{2j} |k(-y)| \, N_{q,\alpha}^+(F; ty) dy dt$$

$$= C \int_0^{1/j} \int_j^{2j} |k(-y)| \, N_{q,\alpha}^+(F; ty) dy dt + C \int_{1/j}^1 \int_j^{2j} |k(-y)| \, N_{q,\alpha}^+(F; ty) dy dt$$

$$= S_1(j) + S_2(j)$$

By (2), it follows that the inner integrals in $S_1(j)$ and $S_2(j)$ are bounded by

$$\left\|N_{q,\alpha}^+(F; .)\right\|_\infty \int_j^{2j} |k(-y)| \, dy \le C_f,$$

and therefore $S_1(j) \to 0$, when $j \to +\infty$. As for $S_2(j)$, we will see that

$$\left[\int_j^{2j} |k(-y)| \, N_{q,\alpha}(F; ty) dy\right] \to 0$$

. Using condition (b) of k, changing variables and by Hölder's inequality, if $s_1 > s \ge 1$ and for $t > 1/j$, we get

$$\int_j^{2j} |k(-y)| \, N_{q,\alpha}^+(F; ty) dy \le \int_{tj<z<2tj} |z|^{-1} N_{q,\alpha}^+(F; z) dz \tag{11}$$

$$\le \left(\int_{tj<z<2tj} N_{q,\alpha}^+(F; z)^{s_1} w(z) dz\right)^{1/s_1} \left(\int_{z>1} \frac{w^{-\frac{s_1'}{s_1}}(z)}{|z|^{s_1'}} dz\right)^{1/s_1'}$$

Since $w^{-\frac{s_1'}{s_1}} \in A_{s_1'}^-$, by the version for M^- of (1), we have that $\int_{z>1} \frac{w^{-\frac{s_1'}{s_1}}(z)}{|z|^{s_1'}} dz \le C_w$, then since

$$\left(\int_{tj<z<2tj} N_{q,\alpha}^+(F; z)^{s_1} w(z) dz\right)^{1/s} \le \left\|N_{q,\alpha}^+(F; .)\right\|_\infty^{\frac{s-p}{s}} \left(\int_{tj<z<2tj} N_{q,\alpha}^+(F; z)^p w(z) dz\right)^{1/s},$$

tends to zero for each $t \ge 0$, we obtain $S_2(j) \to 0$, when $j \to +\infty$. $\square$

Lemma 2.4. *Let $F \in \mathcal{H}_{q,\alpha}^{p,+}(w) \cap \Lambda_\alpha$, and let $f(y)$ be a representative of F. Let k be a one-sided regular Calderón-Zygmund kernel of order $[\alpha]+1$. If we define*

$$g_j(x) = \tag{12}$$

$$v.p. \int k(x-y)f(y)\phi_j(y)dy - \sum_{i=0}^{N} \int D^i k(-y)f(y)(\phi_j - \phi_1(y))dy \frac{x^i}{i!},$$

where $\phi_j(y)$ and $\phi_1(y)$ are given as in (8), then there exists $\lim_{j\to\infty} g_j$ in $L^q_{loc}(x_{-\infty}, \infty)$.

Proof. If we denote f_0 the representative of F such that $n^+_{q,\alpha}(f_0, 0) = N^+_{q,\alpha}(F, 0)$, we have that $f(y) = f_0(y) + P(y)$, where $P(y)$ is a polynomial of degree at most N. Let us fix an interval $I = [a, b] \subset (x_{-\infty}, \infty)$, and we consider a natural number l such that $I \subset [-l/2, l/2]$. Then, for every $x \in I$, and if $j > l$ we have that

$$g_j(x) - g_l(x) = \int [k(x-y) - \sum_{i=0}^{N} \int D^i k(-y)\frac{x^i}{i!}] f(y)(\phi_j(y) - \phi_l(y))dy, \tag{13}$$

We will prove that the limit of the right hand side of (13) exists. We consider two cases, the first when α is not a natural number, i.e., $\alpha = N + \beta$ where $0 < \beta < 1$, and the second when $\alpha = N + 1$. In the first case, if $x \in I \subset [-l/2, l/2]$, since $\text{supp}(1 - \phi_l) \subset |y| \geq l$, by Taylor's formula, (5), Lemma 2.2 and the estimate $|P(y)| \leq C(|y| + 1)^N$, we get the following estimate for the right hand side of (13)

$$\int_{|y|>l} \left| k(x-y) - \sum_{i=0}^{N} D^j k(-y)\frac{x^i}{i!} \right| |f(y)| (1 - \phi_l(y))dy \tag{14}$$

$$\leq \int_{|y|>l} \left|D^{N+1}k(\xi x - y)\right| |f(y)| \, dy \frac{x^{N+1}}{(N+1)!} \leq C_l \int_{|y|>l} |\xi x - y|^{-(N+2)} |f_0(y) + P(y)| \, dy$$

$$\leq C_l \left\|N^+_{q,\alpha}(F; .)\right\|_\infty \int_{|y|>l} |y|^{-(N+2)} |y|^{N+\beta} \, dy + C_l \int_{|y|>l} |y|^{-(N+2)} (|y| + 1)^N dy \ < \infty,$$

Therefore, by Bounded Convergence Theorem the right hand side of (13) converges to

$$\int \left[k(x-y) - \sum_{i=0}^{N} D^j k(-y)\frac{x^i}{i!} \right] f(y)(1 - \phi_l(y))dy.$$

when $j \to \infty$. We observe that in this case, i.e., when $0 < \beta < 1$, it is enough to assume that $F \in \Lambda_\alpha$ to prove the lemma.

In the second case, i.e. $\alpha = N + 1$, in order to show that the limit of the right hand side of (13) exists, we have to consider the cases $f(y) = P(y)$ and $f(y) = f_0(y)$. For the case $f(y) = P(y)$ we argue as before. As for the case $f = f_0$, we can write the right hand side of (13) as

$$\int [k(x-y) - \sum_{i=0}^{N+1} D^j k(-y)\frac{x^i}{i!}] \, f_0(y)(\phi_j(y) - \phi_l(y))dy$$

$$+ \int D^{N+1}k(-y) f_0(y)(\phi_j(y) - \phi_l(y))dy \frac{x^{N+1}}{(N+1)!}. \tag{15}$$

For the first term of (15), proceeding in the same way that for $\beta < 1$, we see that this term converges to

$$\int [k(x-y) - \sum_{i=0}^{N+1} D^j k(-y)\frac{x^i}{i!}] \, f_0(y)(1 - \phi_l(y))dy.$$

Integrating by parts, we obtain that the second term of (15) coincides with

$$(-1)^{N+1}\int k(-y)D^{N+1}\left[f_0(y)(\phi_j(y)-\phi_l(y))\right]dy.$$

By Leibnitz's formula, and since $\mathrm{supp}(k)\subset(-\infty,0)$, the integral above is equal to

$$\begin{aligned}(16)\qquad &\sum_{i=0}^{N+1} C_{N,i}\int_l^{2l} k(-y)D^i f_0(y)D^{N+1-i}(\phi_j(y)-\phi_l(y))dy\\ &+\sum_{i=0}^{N+1} C_{N,i}\int_{y>2l} k(-y)D^i f_0(y)D^{N+1-i}\phi_j(y)dy.\end{aligned}$$

If $j>2l$, the first sum in (16) is equal to

$$\begin{aligned}&\sum_{i=0}^{N} C_{N,i}\int_l^{2l} k(-y)D^i f_0(y)D^{(N+1-i)}\phi_l(y)dy\\ (17)\qquad &+\int_l^{2l} k(-y)D^{N+1}f_0(y)(1-\phi_l(y))dy.\end{aligned}$$

By (2) and Lemma 2.1, the last term is bounded by

$$C\left\|D^{N+1}f_0\right\|_\infty\int_l^{2l}|k(-y)|\,dy\le C\left\|D^{N+1}f_0\right\|_\infty\le C\left\|N_{q,\alpha}^+(F;.)\right\|_\infty.$$

On the other hand, taking into account Lemma 2.2, the inequality $\left|D^{(N+1-i)}\phi_l(y)\right|\le Cl^{-(N+1-i)}$ and (2), we obtain that each term of the sum in (17) is bounded by

$$\begin{aligned}&\int_l^{2l}|k(-y)|\left|D^i f_0(y)\right|\left|D^{(N+1-i)}\phi_l(y)\right|dy\le\\ C\left\|N_{q,\alpha}^+(F;.)\right\|_\infty&\int_l^{2l}|k(-y)|\,|y|^{N+1-i}\ \ l^{-(N+1-i)}dy\le C\left\|N_{q,\alpha}^+(F;.)\right\|_\infty.\end{aligned}$$

As for the second sum in (16), by Lemma 2.3, the terms corresponding to $i<N+1$ converge to zero, and the term $\int_{y>2l}k(-y)D^{N+1}f_0(y)\phi_j(y)dy$ converges to $\int_{y>2l}k(-y)D^{N+1}f_0(y)dy$, in fact the pointwise convergence of the integrand is clear, and by Lemma 2.1, for $s_1>s\ge1$, we have that

$$\begin{aligned}&\int_{|y|>2l}\left|k(-y)D^{N+1}f_0(y)\phi_j(y)\right|dy\le\int_{y>2l}|y|^{-1}\left|D^{N+1}f_0(y)\right|dy\\ &\le\left(\int_{|y|>2l}N_{q,\alpha}^+(F;y)^{s_1}w(y)dy\right)^{1/s_1}\left(\int_{y>2l}|y|^{-s_1'}\,w^{-\frac{s_1'}{s_1}}(y)dy\right)^{1/s_1'}\\ &\le C_{w,l}\left\|N_{q,\alpha}^+(F;.)\right\|_\infty^{\frac{s_1-p}{s_1}}\left(\int_{|y|>2l}N_{q,\alpha}^+(F;y)^p w(y)dy\right)^{1/s_1}<\infty\end{aligned}$$

Then, $\lim\limits_{j\to\infty}g_j(x)$ exists in $L^q_{loc}(x_{-\infty},\infty)$. $\square$

Taking into account the notation of the previous lemma, for $F\in\mathcal{H}_{q,\alpha}^{p,+}(w)\cap\Lambda_\alpha$, if $f(y)$ is a representative of F and k is a one-sided regular Calderón-Zygmund kernel of order

$[\alpha]+1$, we define

$$K_0 f(x) = \lim_{j\to\infty} g_j(x) \tag{18}$$

$$= \lim_{j\to\infty}\left[v.p.\int k(x-y)f(y)\phi_j(y)dy - \sum_{i=0}^{N}\int D^i k(-y)f(y)(\phi_j-\phi_1(y))dy\frac{x^i}{i!}\right],$$

where the limit is taking in the sense of $L^q_{loc}(x_{-\infty},\infty)$. In Lemma 2.4 we have proved that for $x\in I=[a,b]\subset[-l/2,l/2]$,

$$K_0 f(x) = \lim_{j\to\infty} g_j(x) \tag{19}$$

$$= g_l(x) + \int\left[k(x-y) - \sum_{i=0}^{N}\int D^i k(-y)\frac{x^i}{i!}\right] f(y)(1-\phi_l(y))dy,$$

where $g_l(x) = v.p.\int k(x-y)f(y)\phi_l(y)dy - \sum_{i=0}^{N}\int D^i k(-y)f(y)(\phi_l(y)-\phi_1(y))dy\ \frac{x^i}{i!}$.

Lemma 2.5. *Let $P(y)$ a polynomial of degree at most N, and let $k(y)$ be a regular Calderón-Zygmund kernel of order $N+1$, then $K_0P(x)$ coincides with a polynomial of degree at most N in $(x_{-\infty},\infty)$.*

Proof. Without loss of generality, we can assume that $P(y)=y^n$ where $0\le n\le N$. Let us fix a natural number l, and let $x\in[-l/2,l/2]\cap(x_{-\infty},\infty)$. Then, from (19), we have that

$$K_0P(x) = v.p.\int k(x-y)y^n\phi_l(y)dy + \int\left[k(x-y)-\sum_{i=0}^{N}D^i k(-y)\frac{x^i}{i!}\right]y^n(1-\phi_l(y))dy$$

$$+\sum_{i=0}^{N}\int D^i k(-y)\ y^n(\phi_l-\phi_1(y))dy\frac{x^i}{i!} = S_1(x)+S_2(x)+S_3(x), \tag{20}$$

where $S_3(x)$ is a polynomial of degree at most N. Since $k(y)$ is a regular Calderón-Zygmund kernel of order $N+1$ and $y^n\phi_l(y)\in C_0^\infty$, it easy to see that

$$D^{N+1}S_1(x) = \int k(x-y)D^{N+1}[y^n\phi(y)]dy. \tag{21}$$

As for $S_2(x)$, we can derive under the integral sign, in fact for $h=0,1,2,...,N+1$, and $|y|>l$, by Taylor's formula and (5), we obtain that

$$\left|D_x^h[k(x-y)-\sum_{i=0}^{N}D^i k(-y)\frac{x^i}{i!}]\right| \le C\left|D^{N+1}k(\xi x-y)\right||x|^{N+1-h} \le C_l\,|y|^{-N-2},$$

then

$$\int\left|D_x^h[k(x-y)-\sum_{i=0}^{N}D^i k(-y)\frac{x^i}{i!}]\right||y^n(1-\phi_l(y))|\,dy$$

$$\le\ C_l\int_{|y|>l}|y|^{n-N-2}\,dy<\infty.$$

Therefore $D^{N+1}S_2(x) = \int\left(D_x^{N+1}[k(x-y)]\right)\ y^n(1-\phi_l(y))dy$ and integrating by parts, we obtain

$$D^{N+1}S_2(x) = \int\left(D_x^{N+1}[k(x-y)]\right)\ y^n(1-\phi_l(y))dy$$

$$= \int k(x-y)\ D_y^{N+1}[y^n(1-\phi_l(y))]dy = -\int k(x-y)\ D_y^{N+1}[y^n\phi_l(y)]dy,$$

Then, from (20), and since $S_3(x)$ is a polynomial of degree at most N, we have that $D^{N+1}(K_0 P) \equiv 0$, and the conclusion of lemma follows. $\square$

The previous two lemmas enable us to give the following definition:

Definition 2.6. *Let k be a one-sided regular Calderón-Zygmund kernel of order $[\alpha]+1$. Let $F \in \Lambda_\alpha$ and if, in addition, α is a natural number we assume that F also belongs to $\mathcal{H}^{p,+}_{q,\alpha}(w)$. Then, we define $\overline{K}F$ the class in E^q_N of the function*

$$K_0 f(x) = \tag{22}$$

$$\lim_{j\to\infty}\left[v.p.\int k(x-y)f(y)\phi_j(y)dy - \sum_{i=0}^{N}\int D^i k(-y)f(y)(\phi_j-\phi_1(y))dy\frac{x^i}{i!}\right],$$

where $f(y)$ is a representative of F.

This definition makes sense, since by Lemma 2.4 we have that for each representative of F, the limit in (22) exists in the sense of $L^q_{loc}(x_{-\infty},\infty)$ and by Lemma 2.5 the class $\overline{K}F$ does not depend of the representative f of F. Furthermore, if $x_0 \in (x_{-\infty},\infty)$ and if we define

$$K_{x_0} f(x) = \tag{23}$$

$$\lim_{j\to\infty}[v.p.\int k(x-y)f(y)\phi_{x_0,j}(y)dy - \sum_{i=0}^{N}\int D^i k(x_0-y)f(y)(\phi_{x_0,j}(y)-\phi_{x_0,1}(y))dy\frac{(x-x_0)^i}{i!}],$$

where f is a representative of F. Routine computations show that $K_{x_0}f(x)$ differs from $K_0 f(x)$ in a polynomial of degree at most N, and therefore $\overline{K}F$ is also the class of $K_{x_0}f(x)$. For $x \in [a,b] \subset [x_0-l, x_0+l]$, arguing as before in order to obtain (19), it follows that

$$K_{x_0} f(x) = \tag{24}$$

$$g_{x_0,l}(x) + \int\left[k(x-y) - \sum_{i=0}^{N}\int D^i k(x_0-y)\frac{(x-x_0)^i}{i!}\right]f(y)(1-\phi_{x_0 l}(y))dy,$$

where

$$g_{x_0,l}(x) = v.p.\int k(x-y)f(y)\phi_{x_0,l}(y)dy -$$
$$\sum_{i=0}^{N}\int D^i k(x_0-y)f(y)(\phi_{x_0,l}(y)-\phi_{x_0,1}(y))dy\frac{(x-x_0)^i}{i!}.$$

3. Main results

Theorem 3.1. *Let $w \in A^+_s$ and $0 < p \le 1$, such that $(\alpha+1/q)p \ge s > 1$ or $(\alpha+1/q)p > 1$ if $s=1$. Let $\overline{K}$ be the operator associated with a one-sided regular Calderón-Zygmund kernel $k(x)$ of order $[\alpha]+1$ given in the Definition 2.6. Then, $\overline{K}$ can be extended to a bounded operator from $\mathcal{H}^{p,+}_{q,\alpha}(w)$ into $\mathcal{H}^{p,+}_{q,\alpha}(w)$.*

If α is not a natural number, Theorem 3.1 is a consequence of Corollary 1.3 and of the following result:

Theorem 3.2. *Let $F \in \Lambda_\alpha$, where $\alpha = N + \beta$ is not a natural number, i.e., $0 < \beta < 1$. Let $\overline{K}$ be the operator associated with a one-sided regular Calderón-Zygmund kernel $k(x)$ of order $N+1$ given in the Definition 2.6. Then*

$$N^+_{q,\alpha}(\overline{K}F; x) \leq C N^+_{q,\alpha}(F; x) \text{ for all } x \in (x_{-\infty}, \infty),$$

where C is a finite constant not depending on F.

Proof. Let us fix $x_1 \in (x_{-\infty}, \infty)$ and $\rho > 0$. Let $f(y)$ be the representative of F such that $N^+_{q,\alpha}(F; x_1) = n^+_{q,\alpha}(f, x_1)$. Then, for $x \in \left[x_1, x_1 + \frac{\rho}{4}\right]$, from (24) and associating conveniently we have that

$$\begin{aligned} K_{x_1}(f(1 - \phi_{x_1,\rho})(x) &= \sum_{i=0}^{N} \int D^i k(x_1 - y) f(y) \phi_{x_1,1}(y) dy \frac{(x - x_1)^i}{i!} \\ &- \sum_{i=0}^{N} \int D^i k(x_1 - y) f(y) \phi_{x_1,1}(y) \phi_{x_1,\rho}(y) dy \frac{(x - x_1)^i}{i!} \\ &+ \int \left[k(x - y) - \sum_{i=0}^{N} D^i k(x_1 - y) \frac{(x - x_1)^i}{i!} \right] (1 - \phi_{x_1,\rho}(y)) f(y) dy \\ &= Q(x_1, x) - A + B. \end{aligned} \tag{25}$$

The integrals in $Q(x_1, x)$ are finite. In fact, by Lemma (2.2) and since $\operatorname{supp}(k) \subset (-\infty, 0)$, we obtain

$$\int \left|D^i k(x_1 - y)\right| |f(y)| \phi_{x_1,1}(y) dy \leq C \left\| N^+_{q,\alpha}(F; .) \right\|_\infty \int_{x_1}^{x_1+2} |y - x_1|^{-i-1} |y - x_1|^\alpha dy < \infty.$$

Then, $Q(x_1, x)$ is a polynomial of degree at most N. By (5) and taking into account that $\operatorname{supp}(k(x_1 - y)\phi_{x_1,\rho}(y)) \subset [x_1, x_1 + 2\rho]$, we obtain that each term in A is bounded by

$$\begin{aligned} &\int_{x_1}^{x_1+2\rho} \frac{C}{|y - x_1|^{i+1}} |f(y)| dy \rho^i \leq \sum_{j=0}^{\infty} \frac{C}{(2^{-j}\rho)^{i+1}} \int_{x_1+2^{-j}\rho}^{x_1+2^{-j+1}\rho} |f(y)| dy \rho^i \\ &\leq C \sum_{j=0}^{\infty} \frac{(2^{-j+1}\rho)^{\alpha-i}}{(2^{-j+1}\rho)^\alpha} \left(\frac{1}{(2^{-j+1}\rho)} \int_{x_1}^{x_1+2^{-j+1}\rho} |f(y)|^q dy \right)^{1/q} \rho^i \leq C N^+_{q,\alpha}(F; x_1) \rho^\alpha, \end{aligned} \tag{26}$$

As for B, by Taylor's formula, (5) and since $\beta < 1$, we obtain that it is bounded by

$$\begin{aligned} &\left| \int D^{N+1} k(x_1 + \theta(x - x_1) - y)(1 - \phi_{x_1,\rho}(y)) f(y) dy \frac{(x - x_1)^{N+1}}{(N+1)!} \right| \\ &\leq C \int_{x_1+\rho}^{\infty} |y - x_1|^{-N-2} |f(y)| dy \rho^{N+1} \leq C \sum_{j=0}^{\infty} \frac{1}{(2^j\rho)^{N+2}} \int_{x_1+2^j\rho}^{x_1+2^{j+1}\rho} |f(y)| dy \rho^{N+1} \\ &\leq C \sum_{j=0}^{\infty} \frac{(2^j\rho)^{\alpha-(N+1)}}{(2^j\rho)^\alpha} \left(\frac{1}{(2^j\rho)} \int_{x_1}^{x_1+2^{j+1}\rho} |f(y)|^q dy \right)^{1/q} \rho^{N+1} \\ &\leq C \left(\sum_{j=0}^{\infty} (2^j)^{\beta-1} \right) N^+_{q,\alpha}(F; x_1) \rho^\alpha = C N^+_{q,\alpha}(F; x_1) \rho^\alpha, \end{aligned} \tag{27}$$

Them, from (25), (26) and (27), we obtain that for $x \in \left[x_1, x_1 + \frac{\rho}{4}\right]$,

$$|K_{x_1}(f(1 - \phi_{x_1,\rho})(x) - Q(x_1, x)| \leq C N^+_{q,\alpha}(F; x_1) \rho^\alpha. \tag{28}$$

Now, taking into account that $\phi_{x_1,\rho}$ has a bounded support and considering (23), we have that

$$K_{x_1}(f\phi_{x_1,\rho})(x) = \tag{29}$$

$$v.p.\int k(x-y)f(y)\phi_{x_1,\rho}(y))dy + \sum_{i=0}^{N}\int D^i k(x_1-y)f(y)\phi_{x_1,\rho}(y)(1-\phi_{x_1,1}(y))dy\frac{(x-x_1)^i}{i!}.$$

Arguing as in estimate (26), we obtain that the sum in (29) is bounded by $CN^+_{q,\alpha}(F;x_1)\rho^{\alpha}$. As for the first term, since $\operatorname{supp}(k)\subset(-\infty,0)$ and taking into account that the operator K is bounded in L^q, we obtain

$$\begin{aligned}
&\int_{x_1}^{x_1+\rho/4}\left|v.p.\int k(x-y)\chi_{(x_1,\infty)}(y)\, f(y)\phi_{x_1,\rho}(y))dy\right|^q dx \\
\leq\ & C\int\left|\chi_{(x_1,\infty)}(x)\,\phi_{x_1,\rho}(x)\, f(x)\right|^q dx \leq C\int_{x_1}^{x_1+2\rho}|f(x)|^q\,dx \leq CN^+_{q,\alpha}(F;x_1)^q\rho^{\alpha q+1}.
\end{aligned}$$

Thus

$$\int_{x_1}^{x_1+\rho/4}|K_{x_1}(f\phi_{x_1,\rho})(x)|^q\,dx \leq CN^+_{q,\alpha}(F;x_1)^q\rho^{\alpha q+1}. \tag{30}$$

Therefore, from (28) and (30) we obtain that

$$\int_{x_1}^{x_1+\rho/4}|K_{x_1}f(x)-Q(x_1,x)|^q\,dx \leq CN^+_{q,\alpha}(F;x_1)^q\rho^{\alpha q+1},$$

which implies the conclusion of the theorem. □

We observe that Theorem 3.2 gives a proof of the classic result that singular integral operators associated with regular kernels map Λ_α into Λ_α.

As we have already mentioned if α is not a natural number, then Theorem 3.1 is a consequence of Theorem 3.2. If α is a natural number we could prove Theorem 3.1 from the identification between $\mathcal{H}^{p,+}_{q,\alpha}(w)$ and the one-sided Hardy spaces $H^p_+(w)$ (see [5]). However, we give here a direct proof, which follows from Theorem 1.1 and the following lemma:

Lemma 3.3. *Let $w\in A^+_s$ and $0<p\leq 1$, such that $(\alpha+1/q)\,p\geq s>1$ or $(\alpha+1/q)\,p>1$ if $s=1$. Let $\alpha=N+1$, and let $\overline{K}$ be the operator associated with a one-sided regular Calderón-Zygmund kernel $k(x)$ of order $N+2$ given in the Definition 2.6. Then, if A is a p-atom in $\mathcal{H}^{p,+}_{q,\alpha}(w)$, we have that*

$$\left\|\overline{K}A\right\|_{\mathcal{H}^{p,+}_{q,\alpha}(w)}\leq C, \tag{31}$$

where C is a finite constant not depending on A.

Proof. Let $a(y)$ be the representative of A with compact support, such that $\operatorname{supp}(a)\subset I$, where $N^+_{q,\alpha}(A;x)\leq w(I)^{-1/p}$. Without loss of generality we can suppose that $I=[0,r]$. We will prove the following estimates: let $x_1\in(x_{-\infty},\infty)$ then

(i) If $x_1\notin[-2r,r]$,

$$N^+_{q,\alpha}(\overline{K}A;x_1)\leq C\left(M^+\chi_I(x_1)\right)^{\alpha+1/q}w(I)^{-1/p},$$

and

(ii) If $x_1\in[-2r,r]$,

$$N^+_{q,\alpha}(\overline{K}A;x_1)\leq C\left[w(I)^{-1/p}+\left|K^*\left(D^{N+1}a\right)(x_1)\right|\right],$$

where K^* is given in (4).

Let us consider (i). The function $Ka(x) = \lim_{\varepsilon\to 0^+}\int_{|y-x|>\varepsilon} k(x-y)a(y)dy$ is a representative of $\overline{K}A$. Since $\mathrm{supp}(k) \subset (-\infty, 0)$, if $x_1 > r$, we have that $Ka(x) = 0$ for $x \geq x_1$, this implies (i) for $x_1 > r$. Now, we assume that $x_1 < -2r$. We will argue as in the proof of Theorem 3.2. Let us fix $\rho > 0$, and we assume that $x \in \left[x_1, x_1 + \frac{\rho}{4}\right]$, then

$$K(a(1-\phi_{x_1,\rho}))(x) = \int k(x-y)a(y)(1-\phi_{x_1,\rho}(y))dy.$$

By Taylor's formula, we have that

$$K(a(1-\phi_{x_1,\rho})(x) \tag{32}$$

$$= \sum_{i=0}^{N}\int D^i k(x_1-y)a(y)dy\frac{(x-x_1)^i}{i!} - \sum_{i=0}^{N}\int D^i k(x_1-y)a(y)\phi_{x_1,\rho}(y)dy\frac{(x-x_1)^i}{i!}$$
$$+\int D^{N+1}k(x_1+\theta(x-x_1)-y)(1-\phi_{x_1,\rho}(y))a(y)dy\frac{(x-x_1)^{N+1}}{(N+1)!}$$

In the same way as in the proof of Theorem 3.2, we can see that the first sum in the right hand side of (32) is a polynomial of degree at most N, that we denote $Q(x_1, x)$.

We observe that since $n^+_{q,\alpha}(a, -r) = N^+_{q,\alpha}(A, -r) \leq w(I)^{-1/p}$, we have that

$$\int_0^r |a(y)|\, dy \leq C\, r\left(\frac{1}{2r}\int_{-r}^{r}|a(y)|^q\, dy\right)^{1/q} \leq C\, w(I)^{-1/p}r^{\alpha+1}. \tag{33}$$

Let us suppose first that $\rho \geq \frac{|x_1|-r}{2}$, and therefore that $\rho \geq \frac{|x_1|}{4} \geq \frac{r}{2}$. Then, by the condition (5), since $\mathrm{supp}(a(y)) \subset [0, r]$ and (33), we obtain that

$$\left|\int D^i k(x_1-y)a(y)\phi_{x_1,\rho}(y)dy\frac{(x-x_1)^i}{i!}\right| \leq \int_0^r \frac{C}{|x_1-y|^{i+1}}|a(y)|\, dy\rho^i \tag{34}$$
$$\leq C\frac{\rho^i}{|x_1|^{i+1}}\int_0^r |a(y)|\, dy \leq \frac{r^{\alpha+1}}{|x_1|^{i+1}}w(I)^{-1/p}\rho^i \leq \frac{r^{\alpha+1}}{|x_1|^{\alpha+1}}w(I)^{-1/p}\rho^\alpha.$$

Arguing in a similar way, we get

$$\left|\int \left[D^{N+1}k(x_1+\theta(x-x_1)-y)\right](1-\phi_{x_1,\rho}(y))a(y)dy\frac{(x-x_1)^{N+1}}{(N+1)!}\right| \tag{35}$$
$$\leq C\int_0^r |x_1-y|^{-N-2}\left[1-\phi_{x_1,\rho}(y)\right]|a(y)|\, dy\rho^{N+1} \leq C\left(\frac{r}{|x_1|}\right)^{\alpha+1} w(I)^{-1/p}\rho^\alpha.$$

Now, if $\rho < \frac{|x_1|-r}{2}$, since $x_1 < -2r$ we have that $[x_1-2\rho, x_1+2\rho]\cap[0, r] = \emptyset$, This implies that

$$\int D^i k(x_1-y)a(y)\phi_{x_1,\rho}(y)dy = 0. \tag{36}$$

On the other hand, since $\rho < \frac{|x_1|-r}{2}$ and $x \in [x_1, x_1+\frac{\rho}{4}]$, for any $y \in [0, r]$, we have

$$|x_1+\theta(x-x_1)-y| \geq |x_1| - |x-x_1| - r \geq |x_1| - \frac{\rho}{4} - r \geq \frac{|x_1|}{4}.$$

Then, arguing as before, we get

$$\left|\int \left[D^{N+1}k(x_1+\theta(x-x_1)-y)\right](1-\phi_{x_1,\rho}(y))a(y)dy\frac{(x-x_1)^{N+1}}{(N+1)!}\right| \tag{37}$$
$$\leq C\frac{1}{|x_1|^{N+2}}\int_0^r |a(y)|\, dy\rho^{N+1} \leq C\left(\frac{r}{|x_1|}\right)^{\alpha+1} w(I)^{-1/p}\rho^\alpha.$$

Thus, from the estimates (34), (35) , (36) and (37) and since $\frac{r}{|x_1|}<1$, we obtain

$$\frac{1}{\rho^{\alpha q+1}}\int_{x_1}^{x_1+\rho/4}|K(a(1-\phi_{x_1,\rho})(x)-Q(x_1,x)|^q\,dx \le C\left(\frac{r}{|x_1|}\right)^{\alpha q+1} w(I)^{-q/p} \tag{38}$$

If $\rho<\frac{|x_1|-r}{2}$, the supports of $a\,(y)$ and $\phi_{x_1,\rho}\,(y)$ are disjoint and therefore $K(a\phi_{x_1,\rho})(x)=0$. If $\rho\ge\frac{|x_1|-r}{2}\ge\frac{|x_1|}{4}$, since K is bounded on L^q and by (33), we get

$$\frac{1}{\rho^{aq+1}}\int_{x_1}^{x_1+\frac{\rho}{4}}|K(a\phi_{x_1,\rho})(x)|^q\,dx \le \frac{C}{|x_1|^{\alpha q+1}}\int_0^r|a(x)|^q\,dx\le C\frac{r^{\alpha q+1}w(I)^{-q/p}}{|x_1|^{\alpha q+1}}. \tag{39}$$

Then, from (38) and (39) , we obtain

$$\frac{1}{\rho^{a+1/q}}(\int_{x_1}^{x_1+\frac{\rho}{4}}|Ka(x)-Q(x_1,x)|^q\,dx)^{1/q}\le C\left[M^+\chi_I(x_1)\right]^{\alpha+1/q}w(I)^{-1/p},$$

which implies (i).

Now, we prove (ii). Let $x_1\in[-2r,r]$. Let $f(y)\in A$, such that $n^+_{q,\alpha}(f;x_1)=N^+_{q,\alpha}(A;x_1)$. Let $\rho>0$ and $x\in\left[x_1,x_1+\frac{\rho}{4}\right]$. In the proof of Theorem 3.2 we saw that $Q(x_1,x)=\sum_{i=0}^N\int D^ik(x_1-y)f(y)\phi_{x_1,1}(y)dy\frac{(x-x_1)^i}{i!}$ is a polynomial and furthermore

$$\begin{aligned}K_{x_1}(f(1-\phi_{x_1,\rho})(x)-Q(x_1,x)&=-\sum_{i=0}^N\int D^ik(x_1-y)f(y)\phi_{x_1,1}(y)\phi_{x_1,\rho}(y)dy\frac{(x-x_1)^i}{i!}\\&\quad+\int[k(x-y)-\sum_{i=0}^N D^ik(x_1-y)\frac{(x-x_1)^i}{i!}](1-\phi_{x_1,\rho}(y))f(y)dy\\&=I_1+I_2.\end{aligned} \tag{40}$$

Proceeding as in estimate (26) we obtain that $|I_1|$ is bounded by $Cw(I)^{-1/p}\rho^\alpha$. Subtracting and adding $D^{N+1}k(x_1-y)\frac{(x-x_1)^{N+1}}{(N+1)!}$ in the integrand of I_2 and arguing as in estimate (27), we get that

$$|I_2|\le Cw(I)^{-1/p}\rho^\alpha+\left|\int\left[D^{N+1}k(x_1-y)\right](1-\phi_{x_1,\rho}(y))f(y)dy\right|\rho^\alpha. \tag{41}$$

Integrating by parts and by Leibnitz's formula, we have

$$\begin{aligned}\left|\int\left[D^{N+1}k(x_1-y)\right](1-\phi_{x_1,\rho}(y))f(y)dy\right|&=\left|\int k(x_1-y)D^{N+1}\left[(1-\phi_{x_1,\rho}(y))f(y)\right]dy\right|\\&\le\left|\sum_{i=1}^{N+1}C_{N,i}\int k(x_1-y)D^{N+1-i}f(y)D^i\left(1-\phi_{x_1,\rho}(y)\right)dy\right|\\&\quad+\left|\int k(x_1-y)D^{N+1}a(y)\left(1-\phi_{x_1,\rho}(y)\right)dy\right|.\end{aligned} \tag{42}$$

For $i\ge1$, the support of $D^i\left(1-\phi_{x_1,\rho}(y)\right)$ is contained in $\{y:\rho\le|y-x_1|\le2\rho\}$. Then, since $\operatorname{supp}(k)\subset(-\infty,0)$, $\left|D^i\phi_{x_1,\rho}(y)\right|\le C_i\rho^{-i}$ and by Lemma 2.2 we get that the sum in the second line of (42) is bounded by $Cw(I)^{-1/p}$. As for the last summand of (42), using

Lemma 2.1, we obtain that it is bounded by

$$\left|\int_{\rho<|y-x_1|\le 2\rho} |k(x_1-y)| \left|D^{N+1}a(y)\right| dy\right| + \left|\int_{|y-x_1|>2\rho} k(x_1-y)D^{N+1}a(y)dy\right|$$

$$\le Cw(I)^{-1/p}\int_{\rho<|y-x_1|\le 2\rho} |k(x_1-y)| + \sup_{\rho>0}\left|\int_{|y-x_1|>2\rho} k(x_1-y)D^{N+1}a(y)dy\right|$$

$$\le Cw(I)^{-1/p} + K^*(D^{N+1}a)(x_1). \tag{43}$$

Then, from (40), (41) and (43) , we obtain for $x\in\left[x_1,x_1+\frac{\rho}{4}\right]$ that

$$|K_{x_1}(f(1-\phi_{x_1,\rho})(x)-Q(x_1,x)|\le C\left[w(I)^{-1/p}+K^*(D^{N+1}a)(x_1)\right]\rho^{\alpha}.$$

On the other hand, proceeding as in the proof of (30) in Theorem 3.2, we get

$$\int_{x_1}^{x_1+\rho/4} |K_{x_1}(f\phi_{x_1,\rho}(x)|^q\,dx\le Cw(I)^{-q/p}\rho^{\alpha q+1},$$

Thus, we have that

$$\left(\int_{x_1}^{x_1+\rho/4}|K_{x_1}f(x)-Q(x_1,x)|^q\,dx\right)^{1/q}\le C\left[w(I)^{-1/p}+K^*(D^{N+1}a)(x_1)\right]\rho^{\alpha+1/q},$$

which implies (ii).

Finally, we will see that (i) and (ii) imply the lemma. By (i) and (1), we obtain

$$\int_{(x_{-\infty},\infty)\cap x\notin[-2r,r]} N^+_{q,\alpha}(\overline{K}A;x)^p w(x)dx\le Cw(I)^{-1}\int\left[M^+\chi_I(x_1)\right]^{(\alpha+1/q)p}w(x)dx\le C.$$

By (ii), Hölder's inequality and Theorem 1.4, we get that

$$\int_{(x_{-\infty},\infty)\cap x\in[-2r,r]} N^+_{q,\alpha}(\overline{K}A;x)^p w(x)dx\le C_w+\int_{-2r}^{r}K^*(D^{N+1}a)(x_1)^p w(x)dx$$

$$\le\ C_w+\left(\int K^*(D^{N+1}a)(x_1)^{p(N+2)}w(x)dx\right)^{\frac{1}{N+2}}\left(\int_{-2r}^{r}w(x)dx\right)^{1-\frac{1}{N+2}}$$

$$\le\ C_w+C_w\left(\int_0^r\left(D^{N+1}a)(x_1)\right)^{p(N+2)}w(x)dx\right)^{\frac{1}{N+2}}w([-2r,r])^{1-\frac{1}{N+2}}\le C_w,$$

which concludes the proof. □

References

[1] H. Aimar, L. Forzani and F. Martín-Reyes, *On weighted inequalities for singular integrals.* Proc. Amer. Math. Soc. Volumen 125, number 7, 1997, 2057-2064.

[2] A. P. Calderón, *Estimates for singular integral operators in terms of maximal functions.* Studia Math. 44 (1972), 563-582.

[3] A. B. Gatto, J. G. Jiménez and C. Segovia, *On the solution of the equation* $\Delta^m F=f$ *for* $f\in H^p$. Conference on Harmonic Analysis in honor of Antoni Zygmund, Volumen II, Wadsworth international mathematics series, 1983.

[4] F. J. Martín-Reyes, *New proofs of weighted inequalities for the one-sided Hardy-Littlewood maximal functions*, Proc. Amer. Math. Soc. 117 (1993), 691-698.

[5] Sheldy Ombrosi, *On spaces associated with primitives of distributions in one-sided Hardy spaces.* To appear

[6] L. de Rosa and C. Segovia, *Weighted* H^p *spaces for one sided maximal functions,* Contemporary Math., volumen 189, (1995) 161-183.

[7] E. Sawyer, *Weighted inequalities for the one-sided Hardy-Littlewood maximal functions*, Trans. Amer. Math. Soc. 297 (1986), 53-61.

S. Ombrosi.

Depto. de Matemática, Universidad NAcional del Sur, Bahia Blanca, Buenos Aires, Argentina.
e-mail: sombrosi@criba.edu.ar

C. Segovia.
Depto. de Matemática, FCEyN, Univ. de Buenos Aires, Ciudad Universitaria (1428), Buenos Aires, and IAM, CONICET, Argentina.

e-mail: segovia@iamba.edu.ar