SHARP A_1 BOUNDS FOR CALDERÓN-ZYGMUND OPERATORS AND THE RELATIONSHIP WITH A PROBLEM OF MUCKENHOUPT AND WHEEDEN

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Abstract. For any Calderón-Zygmund operator T the following sharp estimate is obtained for $1 < p < \infty$:

$$\|T\|_{L^p(w)} \leq c\,\nu_p\,\|w\|_{A_1},$$

where $\nu_p = \frac{p^2}{p-1}\log\left(e + \frac{1}{p-1}\right)$. In the case when $p = 2$ and T is a classical convolution singular integral, this result is due to R. Fefferman and J. Pipher. Then, we deduce the following weak type $(1,1)$ estimate related to a problem of Muckenhoupt and Wheeden:

$$\sup_{\lambda>0} \lambda w\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\} \leq c\,\varphi(\|w\|_{A_1}) \int_{\mathbb{R}^n} |f|\,w dx,$$

where $w \in A_1$ and $\varphi(t) = t(1 + \log^+ t)(1 + \log^+ \log^+ t)$.

1. Introduction

In 1971, C. Fefferman and E.M. Stein [8] established the following extension of the classical weak-type $(1,1)$ property of the Hardy-Littlewood maximal operator M:

$$\sup_{\lambda>0} \lambda\, w\{x \in \mathbb{R}^n : Mf(x) > \lambda\} \leq c \int_{\mathbb{R}^n} |f|\, Mw dx, \tag{1.1}$$

where a weight w is supposed to be a non-negative locally integrable function and $w(E) = \int_E w(x)dx$. This estimate yields some sort of duality for M. It was used in [8] to derive the vector-valued extension of the classical estimates for the Hardy-Littlewood maximal function which has many important applications.

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Assume now that T is a Calderón-Zygmund singular integral operator. It was conjectured by B. Muckenhoupt and R. Wheeden [11] many years ago that the full analogue of (1.1) holds for T, namely,

$$\sup_{\lambda>0} \lambda w\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\} \le c \int_{\mathbb{R}^n} |f|\, Mw dx. \tag{1.2}$$

Note that the question whether inequality (1.2) holds is still open even for the Hilbert transform. Moreover, it is still unknown whether the following weaker variant of (1.2) is true: if w is an A_1 weight, then

$$\sup_{\lambda>0} \lambda\, w\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\} \le c\|w\|_{A_1} \int_{\mathbb{R}^n} |f| w dx. \tag{1.3}$$

Recall that w is an A_1 weight if there is a finite constant c such that $Mw \le c\, w$ a.e., and where $\|w\|_{A_1}$ denotes the smallest of these c.

In this paper we shall be concerned with inequality (1.3) which can be called as the weak Muckenhoupt and Wheeden conjecture. As far as we know, (1.3) was shown to be true by S. Buckley [1] only for power weights $w_\delta(x) = |x|^{-n(1-\delta)}, 0 < \delta < 1$. Observe that for these weights (1.2) holds as well (since $\|w_\delta\|_{A_1} w_\delta \le cMw_\delta$). However, in the general case the problem seems to be much more complicated.

In order to study inequality (1.3), it is natural to ask first about the dependence of $L^p(w)$ operator norms of T on $\|w\|_{A_1}$ for $p > 1$. We discuss briefly the known results in this direction.

Denote by α the best possible exponent in the inequality

$$\|T\|_{L^p(w)} \le c_{n,p} \|w\|_{A_1}^{\alpha}. \tag{1.4}$$

In the case when $p = 2$ and $T = H$ is the Hilbert transform, R. Fefferman and J. Pipher [7] established that $\alpha = 1$. The proof is based on sharp A_1 bounds for appropriate square functions on $L^2(w)$ from the works [2, 3], in particular, a deep inequality of Chang-Wilson-Wolff was used. One can show that this approach yields $\alpha = 1$ also for $p > 2$. However, for $1 < p < 2$ the same approach gives the estimate $\alpha \le 1/2 + 1/p$. Also, that approach works only for classical singular integrals.

We mention some recent results by S. Petermichl and A. Volberg [15] for the Ahlfors-Beurling transform and by S. Petermichl [13, 14] for the Hilbert transform and the Riesz Transforms. In these papers it has been shown that if T is any of these operators, then

$$\|T\|_{L^p(w)} \leq c_{p,n} \|w\|_{A_p}^{\max\{1,\frac{1}{p-1}\}} \quad (1 < p < \infty), \tag{1.5}$$

and the exponent $\max\{1, \frac{1}{p-1}\}$ is best possible. Here A_p denotes the class of weights for which

$$\|w\|_{A_p} \equiv \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} < \infty.$$

Note that $A_1 \subset A_p$, and $\|w\|_{A_p} \leq \|w\|_{A_1}$. Therefore, (1.5) clearly gives that $\alpha = 1$ in (1.4) when $p \geq 2$. However, (1.5) cannot be used in order to get the sharp exponent α in the range $1 < p < 2$, becoming the exponent worst when p gets close to 1. Note also that the proofs in [13, 14, 15] are based on the Bellman function technique, and it is not clear whether they can be extended to the wider class of Calderón-Zygmund operators.

In this paper we use a different approach to show that for any Calderón-Zygmund operator, the sharp exponent in (1.4) is $\alpha = 1$ for all $1 < p < \infty$. Our method is more closely related to the classical Calderón-Zygmund techniques but refining some known estimates.

We state now our main theorems. From now on T will always denote any Calderón-Zygmund operator (see next section).

Theorem 1.1. *Let $1 < p < \infty$ and let $\nu_p = \frac{p^2}{p-1}\log\left(e + \frac{1}{p-1}\right)$. There is a constant $c = c(n,T)$ such that for any A_1 weight w,*

$$\|T\|_{L^p(w)} \leq c\, \nu_p \, \|w\|_{A_1}. \tag{1.6}$$

The main result related to the weak Muckenhoupt and Wheeden conjecture (1.3) is the following.

Theorem 1.2. *Let $\varphi(t) = t(1 + \log^+ t)(1 + \log^+ \log^+ t)$. There is a constant $c = c(n, T)$ such that for any A_1 weight w and for all $f \in L^1(w)(\mathbb{R}^n)$,*

$$\sup_{\lambda>0} \lambda w\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\} \leq c\,\varphi(\|w\|_{A_1}) \int_{\mathbb{R}^n} |f|\, w dx.$$

It was observed by R. Fefferman and J. Pipher [7] that if an operator T satisfies (1.4) for some p_0, then $\|T\|_{L^p} = O(p^\alpha), p \to \infty$. It is well-known that for Calderón-Zygmund operators, $\|T\|_{L^p} = O(p)$, and this grows is best possible. This shows that estimate (1.6) is sharp in terms of $\|w\|_{A_1}$ for any $1 < p < \infty$. The behavior of the constant ν_p from (1.6) when $p \to 1$ is important for us as well. The size of ν_p is reflected in the function φ from Theorem 1.2. We do not know whether ν_p in (1.6) can be replaced by $\nu_p' = \frac{p^2}{p-1}$. One can easily show that this is really true if the weak Muckenhoupt and Wheeden conjecture holds.

2. Preliminaries

In this section we gather definitions and results, some of them very well-known, that will be used later.

2.1. **Maximal Operator.** Given a locally integrable function f on $\mathbb{R}^n$, the Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

where the supremum is taken over all cubes Q containing the point x.

We shall be using several well-known facts concerning M. First, if $0 < \delta < 1$, then $(Mf)^\delta \in A_1$ (see [6]), and

$$\|(Mf)^\delta\|_{A_1} \leq \frac{c_n}{1-\delta}. \tag{2.1}$$

Second is the Fefferman-Stein inequality [8] saying that for any weight w,

$$\|Mf\|_{L^p(w)} \leq c_n\, p'\, \|f\|_{L^p(Mw)} \quad (1 < p < \infty), \tag{2.2}$$

where as usual p' denotes the dual exponent of p, $p' = \frac{p}{p-1}$.

2.2. **Calderón-Zygmund operators.** A continuos linear operator $T : C_0^\infty(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ is a Calderón-Zygmund operator if T extends to a bounded operator on $L^2(\mathbb{R}^n)$, and whose distributional kernel K coincides away from the diagonal $x = y$ in $\mathbb{R}^n \times \mathbb{R}^n$, with a function K so that

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy$$

whenever $f \in C_0^\infty(\mathbb{R}^n)$ and $x \notin \operatorname{supp}(f)$, and satisfies the standard estimates, namely, the size estimate

$$|K(x,y)| \le \frac{c}{|x-y|^n}$$

and the regularity condition: for some $\varepsilon > 0$,

$$|K(x,y) - K(z,y)| + |K(y,x) - K(y,z)| \le c\frac{|x-z|^\varepsilon}{|x-y|^{n+\varepsilon}},$$

whenever $2|x - z| < |x - y|$.

We shall need the following inequality proved in [10]: there is a constant $c = c(n,T)$ such that for any weight w, any $0 < \delta \le 1$,

$$\int_{\mathbb{R}^n} |Tf|^\delta w dx \le c \int_{\mathbb{R}^n} (Mf)^\delta Mw dx, \tag{2.3}$$

for any function f such that the left hand side is finite. Note that actually this inequality was proved in [10] for $\delta = 1$ but exactly the same proof gives the case $0 < \delta < 1$ as well. Indeed, the proof was based on the combination of two inequalities:

$$\int_{\mathbb{R}^n} |f| w dx \le c_n \int_{\mathbb{R}^n} (M^\#_{\lambda_n} f) Mw dx$$

and

$$M^\#_\lambda(Tf)(x) \le c_{\lambda,n,T} Mf(x),$$

where the operator $M^\#_\lambda f$ is defined by

$$M^\#_\lambda f(x) = \sup_{Q \ni x} \inf_c ((f-c)\chi_Q)^*(\lambda|Q|)$$

(f^* denotes the non-increasing rearrangement). Now, in order to prove (2.3) for $0 < \delta < 1$ it suffices to combine the same inequalities with the fact that

$$M^\#_\lambda(|f|^\delta)(x) \le M^\#_\lambda(f)(x)^\delta \quad (0 < \delta < 1).$$

3. Proofs of main results

In this section we give the proof of the main results. We first start by proving the following lemma related to the well-known reverse Hölder property of the A_1 weights. Indeed, if $w \in A_1$ there are constants $r > 1$ and $c \geq 1$ such that

$$M_r w(x) \leq c\, w(x), \tag{3.1}$$

where as usual $M_r w = (Mw^r)^{1/r}$. We shall need a more precise version of (3.1).

Lemma 3.1. *Let $w \in A_1$, and let $r_w = 1 + \frac{1}{2^{n+1}\|w\|_{A_1}}$. Then*

$$M_{r_w} w(x) \leq 2\,\|w\|_{A_1}\, w(x). \tag{3.2}$$

The classical proofs of this property for any A_p weights produce non linear growth constants.

Proof of Lemma 3.1. We begin as known proofs of (3.1) (cf. [5, 9]). Setting $w_Q = \frac{1}{|Q|}\int_Q w$, we have by the converse weak-type estimate for M (see [16]) that for $\lambda > w_Q$,

$$\int\limits_{\{x\in Q: M_Q^d w(x)>\lambda\}} w(x)dx \leq 2^n \lambda |\{x \in Q : M_Q^d w(x) > \lambda\}|,$$

where M_Q^d is the dyadic maximal operator restricted to a cube Q. Multiplying both parts of this inequality by $\lambda^{\delta-1}$ and then integrating and using Fubini's theorem, we get

$$\int_Q (M_Q^d w)^\delta w dx \leq (w_Q)^\delta \int_Q w dx + \frac{2^n\delta}{\delta+1}\int_Q (M_Q^d w)^{\delta+1} dx$$

Setting here $\delta = \frac{1}{2^{n+1}\|w\|_{A_1}}$, we obtain

$$\frac{1}{|Q|}\int_Q w^{\delta+1} dx \leq \frac{1}{|Q|}\int_Q (M_Q^d w)^\delta w dx \leq 2(w_Q)^{\delta+1},$$

which proves (3.2). $\square$

Lemma 3.2. *Let T be any Calderón-Zygmund operator. There is a constant $c = c(n,T)$ such that for any weight w and for any $p, r \geq 1$, the following a priori estimate holds*

$$\left(\int_{\mathbb{R}^n} \frac{|Tf|^p}{(M_r w)^{p-1}} dx\right)^{1/p} \leq cp\log(1+p)\left(\int_{\mathbb{R}^n} \frac{(Mf)^p}{(M_r w)^{p-1}} dx\right)^{1/p}, \tag{3.3}$$

for any function f such that the left hand side is finite.

Remark 3.3. It is well-known that the weight $(M_rw)^{1-p}$ belongs to the A_∞ class with constant independent of w. Hence, (3.3) is a particular example of Coifman-type estimate (see [4]). However, known proofs applied to this concrete weight give only the constant $C(p) \approx 2^p$ on the right-hand side. Our novel point here is an improvement of the behavior of $C(p)$ to $C(p) = p\log(p+1)$. Observe that it is still unclear for us whether $\log(p+1)$ can be removed.

Proof of Lemma 3.2. Denote the left-hand side of (3.3) by I. For $0 < \alpha < 1$ we have

$$I^\alpha = \left\| \left(\frac{|Tf|}{M_rw} \right)^\alpha \right\|_{L^{p/a}(M_rw)} = \sup_{\|h\|_{L^{(p/\alpha)'}(M_rw)}=1} \int_{\mathbb{R}^n} |Tf|^\alpha h (M_rw)^{1-\alpha} dx.$$

Next, using (2.3) and Hölder's inequality, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} |Tf|^\alpha h (M_rw)^{1-\alpha} dx &\le c \int_{\mathbb{R}^n} (Mf)^\alpha M(h(M_rw)^{1-\alpha}) dx \\ &\le c \left\| \left(\frac{Mf}{M_rw} \right)^\alpha \right\|_{L^{p/a}(M_rw)} \left\| \frac{M(h(M_rw)^{1-\alpha})}{(M_rw)^{1-\alpha}} \right\|_{L^{(p/a)'}(M_rw)}.\end{aligned}$$

Since $1 - (1-\alpha)(p/\alpha)' = \alpha(p-1)/(p-\alpha)$, by (2.1) and (2.2) we get

$$\begin{aligned}\left\| \frac{M(h(M_rw)^{1-\alpha})}{(M_rw)^{1-\alpha}} \right\|_{L^{(p/a)'}(M_rw)} &= \|M(h(M_rw)^{1-\alpha})\|_{L^{(p/a)'}((M_rw)^{\alpha(p-1)/(p-\alpha)})} \\ \le c\frac{p}{\alpha} \left(\frac{1}{1-\alpha(p-1)/r(p-\alpha)} \right)^{1-\frac{\alpha}{p}} &\le c\frac{p}{\alpha} \left(\frac{1}{1-\alpha} \right)^{1-\frac{\alpha}{p}}.\end{aligned}$$

Therefore,

$$I \le \left(\frac{cp}{\alpha} \right)^{1/\alpha} \left(\frac{1}{1-\alpha} \right)^{1/\alpha - 1/p} \left(\int_{\mathbb{R}^n} (Mf)^p \frac{1}{(M_rw)^{p-1}} dx \right)^{1/p}.$$

Choosing now $\alpha = \frac{\log(1+p)}{1+\log(1+p)}$, we get (3.3). □

The following lemma plays a key role in the proof of the main results.

Lemma 3.4. *Let $\nu_p = \frac{p^2}{p-1} \log\left(e + \frac{1}{p-1}\right)$. Then for any $p > 1$ and for $1 < r < 2$,*

$$\|Tf\|_{L^p(w)} \le c\nu_p \left(\frac{1}{r-1} \right)^{1-1/pr} \|f\|_{L^p(M_rw)}, \tag{3.4}$$

where $c = c(n, T)$.

Proof. Let T^* be the adjoint operator of T. Then (3.4) is equivalent to

$$\left\|\frac{T^*f}{M_rw}\right\|_{L^{p'}(M_rw)} \le c\nu_p\Big(\frac{1}{r-1}\Big)^{1-1/pr}\left\|\frac{f}{w}\right\|_{L^{p'}(w)}, \tag{3.5}$$

where $1/p+1/p'=1$. Since T^* is also Calderón-Zygmund operator, by (3.3) we obtain

$$\left\|\frac{T^*f}{M_rw}\right\|_{L^{p'}(M_rw)} \le cp'\log(1+p')\left\|\frac{Mf}{M_rw}\right\|_{L^{p'}(M_rw)} \le c\frac{\nu_p}{p}\left\|\frac{Mf}{M_rw}\right\|_{L^{p'}(M_rw)}. \tag{3.6}$$

Next we note that by Hölder's inequality,

$$\frac{1}{|Q|}\int_Q fw^{-1/p}w^{1/p} \le \left(\frac{1}{|Q|}\int_Q w^r\right)^{1/pr}\left(\frac{1}{|Q|}\int_Q (fw^{-1/p})^{(pr)'}\right)^{1/(pr)'},$$

and hence,

$$(Mf)^{p'} \le (M_rw)^{p'-1}M\Big((fw^{-1/p})^{(pr)'}\Big)^{p'/(pr)'}.$$

From this, by the classical maximal theorem (i.e., by (2.2) with $w\equiv 1$),

$$\begin{aligned}\left\|\frac{Mf}{M_rw}\right\|_{L^{p'}(M_rw)} &\le c\Big(\frac{p'}{p'-(pr)'}\Big)^{1/(pr)'}\left\|\frac{f}{w}\right\|_{L^{p'}(w)}\\ &= c\Big(\frac{rp-1}{r-1}\Big)^{1-1/pr}\left\|\frac{f}{w}\right\|_{L^{p'}(w)} \le cp\Big(\frac{1}{r-1}\Big)^{1-1/pr}\left\|\frac{f}{w}\right\|_{L^{p'}(w)}.\end{aligned}$$

Combining this inequality with (3.6), we get (3.5), and therefore the proof is complete. □

Proof of Theorem 1.1. Setting $r = 1+1/2^{n+1}\|w\|_{A_1}$ in (3.4) and using Lemma 3.1, we get

$$\|Tf\|_{L^p(w)} \le c\nu_p\|w\|_{A_1}\|w\|_{A_1}^{\frac{1}{p}(1-\frac{1}{r})}\|f\|_{L^p(w)}.$$

It remains to notice that

$$\|w\|_{A_1}^{\frac{1}{p}(1-\frac{1}{r})} \le \|w\|_{A_1}^{\frac{1}{2^{n+1}p\|w\|_{A_1}}} \le e^{\frac{1}{2^{n+1}pe}}.$$

□

Proof of Theorem 1.2. The proof mainly follows the same lines as the proof of Theorem 1.6 in [12], and therefore we omit some details.

Applying the Calderón-Zygmund decomposition to f and λ, we get a family of pairwise disjoint cubes $\{Q_j\}$ such that $\lambda < |f|_{Q_j} \le 2^n\lambda$. Let $\Omega = \cup_j Q_j$, and $\widetilde{\Omega} = \cup_j 2Q_j$. Next, let $f = g + b$, where $g = \sum_j f_{Q_j}\chi_{Q_j}(x) + f(x)\chi_{\Omega^c}(x)$. Then

$$\begin{aligned} w\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\} &\le w(\widetilde{\Omega}) + w\{x \in (\widetilde{\Omega})^c : |Tb(x)| > \lambda/2\} \\ &+ w\{x \in (\widetilde{\Omega})^c : |Tg(x)| > \lambda/2\}. \end{aligned}$$

For the first two terms we have the following estimate (see [12, p. 303])

$$\begin{aligned} w(\widetilde{\Omega}) + w\{x \in (\widetilde{\Omega})^c : |Tb(x)| > \lambda/2\} &\le \frac{c}{\lambda}\int_{\mathbb{R}^n} |f|\, Mw dx \\ &\le \frac{c\|w\|_{A_1}}{\lambda}\int_{\mathbb{R}^n} |f|\, w dx. \end{aligned} \tag{3.7}$$

Now, by (3.4), for any $p > 1$ we have

$$\begin{aligned} & w\{x \in (\widetilde{\Omega})^c : |Tg(x)| > \lambda/2\} \\ & \le c(\nu_p)^p \Big(\frac{1}{r-1}\Big)^{p-1/r} \frac{1}{\lambda^p}\int_{\mathbb{R}^n} |g|^p M_r(w\chi_{(\widetilde{\Omega})^c})dx \\ & \le c(\nu_p)^p \Big(\frac{1}{r-1}\Big)^{p-1/r} \frac{1}{\lambda}\int_{\mathbb{R}^n} |g|\, M_r(w\chi_{(\widetilde{\Omega})^c})dx. \end{aligned}$$

Arguing exactly as in [12, p. 303], we obtain

$$\int_{\mathbb{R}^n} |g| M_r(w\chi_{(\widetilde{\Omega})^c})dx \le c\int_{\mathbb{R}^n} |f| M_r w dx.$$

Combining this estimate with the previous one, and then taking $r = 1 + 1/2^{n+1}\|w\|_{A_1}$, by Lemma 3.1 we get

$$w\{x \in (\widetilde{\Omega})^c : |Tg(x)| > \lambda/2\} \le \frac{c(\nu_p\|w\|_{A_1})^p}{\lambda}\int_{\mathbb{R}^n} |f| w dx.$$

Setting here $p = 1 + \frac{1}{\log(1+\|w\|_{A_1})}$ gives

$$w\{x \in (\widetilde{\Omega})^c : |Tg(x)| > \lambda/2\} \le \frac{c\varphi(\|w\|_{A_1})}{\lambda}\int_{\mathbb{R}^n} |f| w dx.$$

This estimate combined with (3.7) completes the proof. $\square$

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