A BOUNDEDNESS CRITERION FOR GENERAL MAXIMAL OPERATORS

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ABSTRACT. We consider maximal operators $M_{\mathcal{B}}$ with respect to a basis $\mathcal{B}$. In the case when $M_{\mathcal{B}}$ satisfies a reversed weak type inequality, we obtain a boundedness criterion for $M_{\mathcal{B}}$ on an arbitrary quasi-Banach function space X. Being applied to specific $\mathcal{B}$ and X this criterion yields new and short proofs of a number of well-known results. Our principal application is related to an open problem on the boundedness of the two-dimensional one-sided maximal function M^+ on L^p_w.

1. INTRODUCTION

For any point $x \in \mathbb{R}^n$ denote by $\mathcal{B}(x)$ a family of bounded measurable sets of positive measure. The unified collection $\mathcal{B} = \cup_{x\in\mathbb{R}^n}\mathcal{B}(x)$ is called a basis (see [8] and also [9] for a somewhat different definition). For a locally integrable function f on $\mathbb{R}^n$ the Hardy-Littlewood maximal operator associated with $\mathcal{B}$ is defined by

$$M_{\mathcal{B}}f(x) = \sup_{B\in\mathcal{B}(x)} \frac{1}{|B|}\int_B |f(y)|dy.$$

The basis formed by all cubes Q containing x with sides parallel to the axes we denote by $\mathcal{Q}$. If $x = (x_1, \dots, x_n)$ and $\mathcal{B}(x) = \{\prod_{i=1}^n (x_i, x_i + h)\}_{h>0}$, the corresponding basis is denoted by $\mathcal{Q}^+$. The maximal operators associated with $\mathcal{Q}$ and $\mathcal{Q}^+$ are denoted by M and M^+, respectively.

The Hardy-Littlewood maximal operator in its various forms plays a fundamental role in Harmonic Analysis, and its different aspects have been studied in a great number of papers. The most typical problem of interest can be described briefly as follows: given a function space X and a basis $\mathcal{B}$, find a necessary and sufficient condition yielding the boundedness of $M_{\mathcal{B}}$ on X.

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Let $M_{\mathcal{B},r}f = (M_{\mathcal{B}}|f|^r)^{1/r}$. By Hölder's inequality, $M_{\mathcal{B},r}f \le M_{\mathcal{B},s}f$ if $r < s$. In a recent paper [13], the authors established that M is bounded on a quasi-Banach function space X iff M_r is bounded on X for some $r > 1$. For many particular spaces X this self-improving phenomenon was observed before but each case required its own proof. In this paper we complement this result by extending it to a wide class of $\mathcal{B}$ and by obtaining a similar characterization in terms of $M_{\mathcal{B},r}$ for $r < 1$. The case $r > 1$ in [13] was treated by means of the concept of generalized Boyd indices. Here we give a unified and simple approach to both cases $r > 1$ and $r < 1$ using the well-known Rubio de Francia algorithm.

The following definition expresses the relevant property of a basis needed for our purposes. In the case when $\mathcal{B} = \mathcal{Q}$ it was obtained by E.M. Stein [23].

Definition 1.1. We say that a basis $\mathcal{B}$ satisfies the Stein property if there exists a constant $c > 0$ such that for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, for all $B \in \mathcal{B}(x)$ and $\lambda > M_{\mathcal{B}}f(x)$ we have

$$\int_{\{y\in B:|f(y)|>\lambda\}} |f(y)|dy \le c\lambda|\{y \in B : M_{\mathcal{B}}f(y) > \lambda\}|. \tag{1.1}$$

One of our main results is the following.

Theorem 1.2. *Let $X(\mathbb{R}^n)$ be an arbitrary quasi-Banach function space. Suppose $\mathcal{B}$ satisfies Stein's property. Then the following conditions are equivalent:*

(i) $\lim\limits_{\varepsilon\to 0} \varepsilon\|M_{\mathcal{B},1-\varepsilon}\|_X = 0$;
(ii) $M_{\mathcal{B}}$ *is bounded on* X;
(iii) $M_{\mathcal{B},r}$ *is bounded on* X *for some* $r > 1$.

In order to feel the theorem better, let us consider the case when X is the weighted Lebesgue space L^p_w, where a weight w is supposed to be a non-negative locally integrable function. First of all, we have the following.

Corollary 1.3. *Let $\mathcal{B}$ satisfy Stein's property, and let $1 < p < \infty$. If $M_{\mathcal{B}}$ maps L^p_w into $L^{p,\infty}_w$, then $M_{\mathcal{B}}$ actually maps L^p_w into L^p_w.*

Indeed, if $M_{\mathcal{B}} : L^p_w \to L^{p,\infty}_w$, then by the Marcinkiewicz interpolation theorem (see, e.g., [5, p. 29]), $\|M_{\mathcal{B}}\|_{L^q_w} \le c(q-p)^{-1/q}$ for $q > p$. Taking $q = \frac{p}{1-\varepsilon}$, we get $\|M_{\mathcal{B},1-\varepsilon}\|_{L^p_w} \le c\varepsilon^{-1/p}$. It remains to apply (i) $\Rightarrow$ (ii).

Corollary 1.3 shows that in the case when $\mathcal{B}$ satisfies Stein's property, the weak type (p, p) (with respect to w) of $M_{\mathcal{B}}$ is equivalent to the strong type (p,p) for $p>1$. However, the weak type (p,p) property is usually much easier to prove. Consider, for example, the classical maximal operator M. We recall that a weight w satisfies the A_p condition if there exists $c>0$ such that for any cube Q,

$$\Big(\int_Q w\Big)\Big(\int_Q w^{-1/(p-1)}\Big)^{p-1}\le c|Q|^p.$$

By a fundamental theorem of B. Muckenhoupt [17] (see also [4]), M is bounded on L^p_w iff $w\in A_p$. The first proofs of this result [4, 17] depended on a deep property of A_p weights saying that the A_p condition implies $A_{p-\varepsilon}$ for some $\varepsilon>0$. Later, other proofs (see, e.g., [9]), avoiding this property, were found. We now observe that Theorem 1.2 implies easily both Muckenhoupt's theorem and the implication $A_p\Rightarrow A_{p-\varepsilon}$. Indeed, Hölder's inequality along with the A_p condition yields $Mf(x)^p\le cM_w(|f|^p)(x)$ (M_w is the weighted maximal operator), and since any A_p weight is doubling, by a classical covering argument we get the weighted weak type (p,p) of M. This, by Corollary 1.3, proves Muckenhoupt's theorem (only the sufficiency part in this theorem is non-trivial). Next, we clearly have that $M_r:L^p_w\to L^p_w$ for some $r>1$ iff $M:L^{p-\varepsilon}_w\to L^{p-\varepsilon}_w$ for some $\varepsilon>0$. Therefore, by (ii) $\Rightarrow$ (iii) of Theorem 1.2 we get $A_p\Rightarrow A_{p-\varepsilon}$.

Consider now the maximal operator M^+. Given a cube $Q=\prod_{i=1}^n(a_i-h,a_i)$, set $Q^+=\prod_{i=1}^n(a_i,a_i+h)$. We say that a weight w satisfies the A_p^+ condition if there exists $c>0$ such that for any cube Q,

$$\Big(\int_Q w\Big)\Big(\int_{Q^+} w^{-1/(p-1)}\Big)^{p-1}\le c|Q|^p.$$

Only fourteen years after Muckenhoupt's result E. Sawyer [21] proved that in the one-dimensional case M^+ is bounded on L^p_w iff $w\in A_p^+$. The proof in [21] was based on certain Hardy-type inequalities. Later, F.J. Martín-Reyes [14] found another proof in spirit of the classical case of M. Namely, first an equivalence of A_p^+ and the weak-type (p,p) of M^+ was established (which was done in a simple and clever way), and then the property $A_p^+\Rightarrow A_{p-\varepsilon}^+$ was proved. Observe that in Sawyer's work [21] it was already mentioned that the basis $\mathcal{Q}^+$ in the case $n=1$ satisfies Stein's property. Therefore, using only the weak-type (p,p) of M^+ we have, exactly as above, both Sawyer's theorem and the property $A_p^+\Rightarrow A_{p-\varepsilon}^+$.

It turns out that the case $n\ge 2$ in the study of M^+ is much more complicated. In fact, the question whether the full analogue of Sawyer's theorem holds when $n\ge 2$ is still open. Only in a recent paper [7], the authors overcame considerable technical difficulties and proved that in the case $n = 2$ the A_p^+ condition is equivalent to the weak type (p, p) property of M^+. Observe that a dyadic variant of this result was recently obtained in [19] in any dimension. However, the usual, non-dyadic case requires much more delicate analysis, and it is unknown for us whether the covering argument found in [7] in the case $n = 2$ can be extended to $n \geq 3$.

Once an equivalence between the weak type (p, p) of M^+ and the A_p^+ condition is established, it is natural to ask whether the basis $\mathcal{Q}^+, n = 2$, satisfies Stein's property, as in the one-dimensional case. Unfortunately, this is not true as the following example shows.

Example 1.4. *Let $n = 2$. Then $\mathcal{Q}^+$ does not satisfy Stein's property.*

Let $Q_0 = (0,1)^2$ and $f_\varepsilon = \frac{1}{\varepsilon^2}\chi_{(0,\varepsilon)\times(1-\varepsilon,1)}$ for small ε. It is easy to see that $M^+ f_\varepsilon(0) = 1$ and $\{y \in Q_0 : M^+ f_\varepsilon(y) > \lambda\} \subset (0,\varepsilon) \times (0,1)$. Hence, setting in (1.1) $f = f_\varepsilon$ and $B = Q_0$, for any fixed λ such that $1 < \lambda < \frac{1}{\varepsilon^2}$ we get that the left-hand side of (1.1) is equal to 1, while the right-hand side is bounded by $c\lambda\varepsilon$.

Roughly speaking, Theorem 1.2 contains implicitly a large part of standard technique needed to work with "good" maximal operators. The above example shows that this technique falls down when we deal with M^+ in the multi-dimensional case. Nevertheless, some indirect variants of ideas used in proving Theorem 1.2 combined with the above mentioned weak type result for M^+ proved in [7] allows us to get a strong type result for a family of maximal operators closely related to M^+. This family is defined as follows. Given $x = (x_1, x_2)$ and $r \in [0, 1)$, let $Q_{x,h}^r = \prod_{i=1}^2 (x_i + rh, x_i + h)$. For $f \in L^1_{\text{loc}}(\mathbb{R}^2)$ define the maximal operator N_r^+ by

$$N_r^+ f(x) = \sup_{h>0} \frac{1}{|Q_{x,h}^r|} \int_{Q_{x,h}^r} |f(y)| dy.$$

Observe that $N_0^+ f = M^+ f$ and $N_{r_2}^+ f \leq c N_{r_1}^+ f$ for $0 \leq r_1 < r_2 < 1$.

The second main result of this paper is the following.

Theorem 1.5. *Let $1 < p < \infty$. If $w \in A_p^+(\mathbb{R}^2)$, then*

$$\|N_r^+ f\|_{L^p_w} \leq c\|f\|_{L^p_w} \quad (0 < r < 1),$$

where the constant c depends only on w, p and r.

It is easy to show that in the one-dimensional case $N_r^+ f$ is equivalent to $M^+ f$ (see, e.g., [16, Prop. 2.4]), and this is not true in general when $n \geq 2$. Hence, Theorem 1.5 can be regarded as an extension of Sawyer's theorem to the case $n = 2$. Notice that the main question whether the

$A_p^+(\mathbb{R}^2)$ condition is sufficient for the boundedness of M^+ on $L_w^p(\mathbb{R}^2)$ remains open. However, Theorem 1.5 shows that this really holds for an arbitrary big portion of M^+. This gives an additional indication that an answer to the above question should be positive.

The paper is organized as follows. Section 2 contains the proof of Theorem 1.2. Theorem 1.5 is proved in Section 3. Finally, in Section 4 we consider some other applications of Theorem 1.2.

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2. Proof of Theorem 1.2

For the definition of Banach function norm we refer to [2, p. 2]. If the triangle inequality in this definition is replaced by $\|f+g\| \le c(\|f\|+\|g\|)$ for some $c \ge 1$, we get a quasi-norm. A complete quasi-normed space is called a quasi-Banach space. We shall use the following version of the Aoki-Rolewicz theorem (see, e.g., [11, p. 3]) saying that for a quasi-Banach space X,

$$\Big\|\sum_{k=0}^{\infty} f_k\Big\|_X \le 4^{1/\rho}\Big(\sum_{k=0}^{\infty}\|f_k\|_X^{\rho}\Big)^{1/\rho}, \tag{2.1}$$

where $0<\rho\le 1$ is given by $c=2^{1/\rho-1}$ (c is the “quasi-norm” constant).

We say that a weight w satisfies the $A_1(\mathcal{B})$ condition if there exists $c>0$ such that

$$M_{\mathcal{B}}w(x) \le cw(x) \quad \text{a.e.} \tag{2.2}$$

The smallest possible c in (2.2) is denoted by $\|w\|_{A_1(\mathcal{B})}$.

Lemma 2.1. *Suppose $\mathcal{B}$ satisfies Stein's property. If $w \in A_1(\mathcal{B})$, then*

$$M_{\mathcal{B},r}w(x) \le 2\|w\|_{A_1(\mathcal{B})}w(x) \quad \textit{a.e.}, \tag{2.3}$$

where $r = 1 + \frac{\xi}{\|w\|_{A_1(\mathcal{B})}}$, and ξ depends only on the constant c from Definition 1.1.

Remark 2.2. When $\mathcal{B}=\mathcal{Q}$ this lemma was used in a recent paper [12] in order to get some sharp weighted inequalities for singular integrals. Note that actually the lemma is contained implicitly in [4, 9] but the dependence of r on $\|w\|_{A_1(\mathcal{B})}$ is not written there explicitly. Since this point will be important for us, we give a complete proof of the lemma, although the case of general $\mathcal{B}$ is treated exactly as $\mathcal{Q}$.

Proof of Lemma 2.1. Given $B \in \mathcal{B}(x)$, set $E(B) = \{y \in B : w(y) > M_{\mathcal{B}}w(x)\}$. Using Stein's property and Fubini's theorem, we have

$$\begin{aligned}\int_{E(B)} w^{1+\delta}dy &= \delta\int_{M_{\mathcal{B}}w(x)}^{\infty}\lambda^{\delta-1}\int_{\{y\in B:w(y)>\lambda\}}w(y)dyd\lambda\\ &\le c\delta\int_{M_{\mathcal{B}}w(x)}^{\infty}\lambda^{\delta}|\{y\in B: M_{\mathcal{B}}w(y)>\lambda\}|d\lambda\\ &\le \frac{c\delta}{1+\delta}\int_B (M_{\mathcal{B}}w)^{1+\delta}dy \le \frac{c\delta\|w\|_{A_1(\mathcal{B})}^{1+\delta}}{1+\delta}\int_B w^{1+\delta}dy.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_B w^{1+\delta}dy &= \int_{E(B)} w^{1+\delta}dy + \int_{B\setminus E(B)} w^{1+\delta}dy\\ &\le \frac{c\delta\|w\|_{A_1(\mathcal{B})}^{1+\delta}}{1+\delta}\int_B w^{1+\delta}dy + |B|M_{\mathcal{B}}w(x)^{1+\delta}.\end{aligned}$$

Setting $\delta = \frac{1}{3\max(c,1)}\frac{1}{\|w\|_{A_1(\mathcal{B})}}$, we get $\frac{c\delta\|w\|_{A_1(\mathcal{B})}^{1+\delta}}{1+\delta} \le \frac{1}{3}e^{1/3e} \le \frac{1}{2}$, and thus

$$\frac{1}{|B|}\int_B w^{1+\delta}dy \le 2M_{\mathcal{B}}w(x)^{1+\delta}.$$

This proves the lemma with $r = 1+\delta$. □

Proof of Theorem 1.2. Following the Rubio de Francia idea [20], for $0 < \varepsilon < 1$ set

$$R_\varepsilon f(x) = \sum_{k=0}^{\infty}\varepsilon^k M_{\mathcal{B}}^k f(x),$$

where $M_{\mathcal{B}}^k$ is the operator $M_{\mathcal{B}}$ iterated k times and $M_{\mathcal{B}}^0 f = |f|$. Note that $R_\varepsilon f(x) \in A_1(\mathcal{B})$ with $\|R_\varepsilon f\|_{A_1(\mathcal{B})} \le \frac{1}{\varepsilon}$. Also we trivially have $|f| \le R_\varepsilon f$. Therefore, setting $w(x) = R_\varepsilon f(x)$ in (2.3) and using Hölder's inequality, we get

$$M_{\mathcal{B},1+\xi\varepsilon}f(x) \le \frac{2}{\varepsilon}R_\varepsilon f(x) \quad (0<\varepsilon<1). \tag{2.4}$$

Observe that only two implications in Theorem 1.2 are non-trivial, namely, (i) ⇒ (ii) and (ii) ⇒ (iii). To prove the last implication, we apply (2.1) and (2.4) with $\varepsilon < 1/\|M_{\mathcal{B}}\|_X$. Then

$$\begin{aligned}\|M_{\mathcal{B},1+\xi\varepsilon}f\|_X \le \frac{2}{\varepsilon}\|R_\varepsilon f\|_X &\le \frac{2}{\varepsilon}4^{1/\rho}\Big(\sum_{k=0}^{\infty}(\varepsilon^k\|M_{\mathcal{B}}^k f\|_X)^\rho\Big)^{1/\rho}\\ &\le \frac{2}{\varepsilon}4^{1/\rho}\Big(\sum_{k=0}^{\infty}(\varepsilon\|M_{\mathcal{B}}\|_X)^{\rho k}\Big)^{1/\rho}\|f\|_X,\end{aligned}$$

and thus we have (iii) with $r = 1 + \xi\varepsilon$.

The proof of (i) $\Rightarrow$ (ii) is similar. Given $\varepsilon > 0$, set $\nu_\varepsilon = 1 + \xi\varepsilon$. Using (i), fix an $\varepsilon > 0$ such that $\varepsilon\|M_{\mathcal{B},1/\nu_\varepsilon}\|_X < 1$. Denote by X_ε the quasi-Banach space with norm

$$\|f\|_{X_\varepsilon} = \||f|^{\nu_\varepsilon}\|_X^{\frac{1}{\nu_\varepsilon}}.$$

Rewriting (2.4) as

$$M_{\mathcal{B}}f(x) \le \Big(\frac{2}{\varepsilon}R_\varepsilon(|f|^{\frac{1}{\nu_\varepsilon}})(x)\Big)^{\nu_\varepsilon}$$

and applying (2.1) to $X = X_\varepsilon$ (with the corresponding constant $\rho = \rho_\varepsilon$), we get

$$\begin{aligned}
\|M_{\mathcal{B}}f\|_X &\le (2/\varepsilon)^{\nu_\varepsilon}\|R_\varepsilon(|f|^{\frac{1}{\nu_\varepsilon}})\|_{X_\varepsilon}^{\nu_\varepsilon} \\
&\le (2/\varepsilon)^{\nu_\varepsilon}4^{\nu_\varepsilon/\rho_\varepsilon}\Big(\sum_{k=0}^{\infty}(\varepsilon^k\|M_{\mathcal{B}}^k(|f|^{\frac{1}{\nu_\varepsilon}})\|_{X_\varepsilon})^{\rho_\varepsilon}\Big)^{\nu_\varepsilon/\rho_\varepsilon} \\
&= (2/\varepsilon)^{\nu_\varepsilon}4^{\nu_\varepsilon/\rho_\varepsilon}\Big(\sum_{k=0}^{\infty}\varepsilon^{\rho_\varepsilon k}\|M_{\mathcal{B},1/\nu_\varepsilon}^k f\|_X^{\rho_\varepsilon/\nu_\varepsilon}\Big)^{\nu_\varepsilon/\rho_\varepsilon} \\
&\le (2/\varepsilon)^{\nu_\varepsilon}4^{\nu_\varepsilon/\rho_\varepsilon}\Big(\sum_{k=0}^{\infty}(\varepsilon\|M_{\mathcal{B},1/\nu_\varepsilon}\|_X)^{\rho_\varepsilon k}\Big)^{\nu_\varepsilon/\rho_\varepsilon}\|f\|_X.
\end{aligned}$$

We have obtained (ii), and therefore the theorem is proved. □

3. Proof of Theorem 1.5

We first introduce some notation. Given a square $Q = (a, a+h) \times (b, b+h)$, for $\xi > 0$ set $\widetilde{Q}_\xi = (a - \xi h, a+h) \times (b - \xi h, b+h)$ and $Q_\xi^- = (a - \xi h, a) \times (b - \xi h, b)$ (see Figure 1). Let $Q^- = Q_1^-$. Denote $f_Q = \frac{1}{|Q|}\int_Q f$. Let ℓ_Q be the side length of Q. For a measurable set E, let $w(E) = \int_E w$.

Figure 1. $\widetilde{Q}_\xi$ and Q_ξ^-.

As we mentioned in the Introduction, the proof of Theorem 1.5 contains some variants of ideas used in proving Theorem 1.2. The following lemma represents an analogue of Lemma 2.1.

Lemma 3.1. *There exists a constant $c>0$ such that for any weight w and for any square Q,*

$$\int_Q w^{1+\delta}\le c\frac{\delta}{\xi^2}\int_{\widetilde{Q}_\xi}(M^+w)^{1+\delta}+|Q|(w_Q)^{1+\delta}\quad(\delta>0,0<\xi\le 1).$$

Proof. By Stein's estimate [23], for $\lambda>w_Q$,

$$\int_{\{x\in Q:w(x)>\lambda\}}w(x)dx\le 4\lambda|\{x\in Q:M_Q^{\Delta}w(x)>\lambda\}|,$$

where M_Q^{Δ} is the dyadic maximal function restricted to a square Q. From this, by Fubini's theorem we have,

$$\begin{aligned}\int_{\{x\in Q:w(x)>w_Q\}}w^{1+\delta}dx &= \delta\int_{w_Q}^{\infty}\lambda^{\delta-1}\int_{\{x\in Q:w(x)>\lambda\}}w(x)dxd\lambda\\ &\le 4\delta\int_{w_Q}^{\infty}\lambda^{\delta}|\{x\in Q:M_Q^{\Delta}w(x)>\lambda\}|d\lambda. \qquad (3.1)\end{aligned}$$

Let us show now that for $\lambda>w_Q$ and $0<\xi\le 1$,

$$|\{x\in Q:M_Q^{\Delta}w(x)>\lambda\}|\le\frac{c}{\xi^2}|\{x\in\widetilde{Q}_\xi:M^+w(x)>\lambda/4\}|. \qquad (3.2)$$

We have that $\{x\in Q:M_Q^{\Delta}w(x)>\lambda\}=\cup_jQ_j$, where $w_{Q_j}>\lambda$. For any point $x\in(Q_j)_\xi^-$ there exists a square Q_j' containing Q_j with $|Q_j'|\le 4|Q_j|$, and such that x is the lower left corner of Q_j'. It follows from this that $w_{Q_j'}\ge\frac14 w_{Q_j}>\frac{\lambda}{4}$. Therefore, $M^+w(x)>\frac{\lambda}{4}$ for all $x\in(Q_j)_\xi^-$. Next, we note that $Q_j\subset(1+\frac{2}{\xi})(Q_j)_\xi^-$. Applying the Vitali covering lemma (see, e.g., [2, p. 118]) to the family $\{(1+\frac{2}{\xi})(Q_j)_\xi^-\}$ we get pairwise disjoint squares $(1+\frac{2}{\xi})(Q_i)_\xi^-, i=1,\dots,k$ such that

$$\begin{aligned}\Big|\bigcup_j Q_j\Big| &\le \Big|\bigcup_j(1+\frac{2}{\xi})(Q_j)_\xi^-\Big|\\ &\le 16\sum_{i=1}^k\big|\big(1+\frac{2}{\xi}\big)(Q_i)_\xi^-\big|=16\big(1+\frac{2}{\xi}\big)^2\sum_{i=1}^k|(Q_i)_\xi^-|. \qquad (3.3)\end{aligned}$$

Next we clearly have that the squares $(Q_i)_\xi^-, i=1,\dots,k$ are also pairwise disjoint, and $\cup_{i=1}^k(Q_i)_\xi^-\subset\{x\in\widetilde{Q}_\xi:M^+w(x)>\lambda/4\}$. From this and from (3.3) we get (3.2).

Applying (3.1) and (3.2) gives

$$\int_{\{x\in Q: w(x)>w_Q\}} w^{1+\delta}dx \le c\frac{\delta}{\xi^2}\int_{\widetilde{Q}_\xi}(M^+w)^{1+\delta}dx,$$

from which the lemma follows easily. $\square$

The next lemma will be an important ingredient in proving of the subsequent statement.

Lemma 3.2. *Let F be the convex hull of $Q_\xi^-\cup Q, \xi\ge 1$ (see Figure 2), and let $w\in A_p^+$. Then*

$$w(F)\le cw(Q),$$

where the constant c depends only on ξ, p and w.

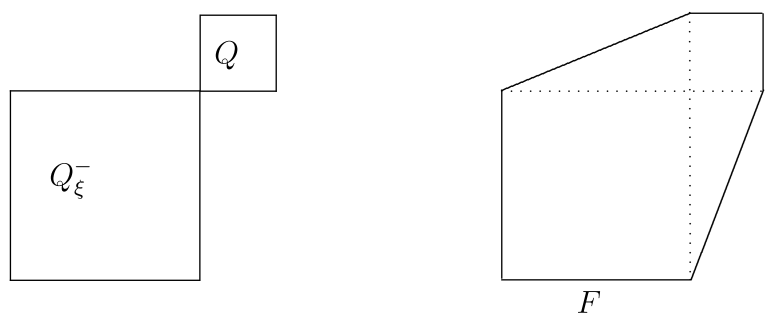

FIGURE 2. Convex hull

Proof. When $\xi = 1/4$ this was proved by F.J. Martín-Reyes [15]. In general case the proof is similar but we give it for the sake of completeness.

We observe first that for any square Q,

$$w(Q_\xi^-)\le cw(Q). \tag{3.4}$$

Indeed, note that $Q\subset (Q_\xi^-)^+$. Therefore, setting $\sigma = w^{-1/(p-1)}$ and applying the A_p^+ condition along with Hölder's inequality, we get

$$w(Q_\xi^-)\sigma\big((Q_\xi^-)^+\big)^{p-1}\le c\xi^p|Q|^p\le c\xi^p w(Q)\sigma\big((Q_\xi^-)^+\big)^{p-1},$$

which proves (3.4).

Next we have that $F\setminus(Q_\xi^-\cup Q)$ is the union of two triangles $T_1\cup T_2$. In view of (3.4), it remains to show that $w(T_i)\le cw(Q), i=1,2$. By symmetry, it suffices to consider the case $i=1$.

Let $Q = (a, a+h) \times (b, b+h)$. Then it is easy to see that T_1 is covered (up to a set of measure zero) by $\cup_{j=0}^{\infty} Q_j$, where

$$Q_j = \Big(a - \frac{\xi h}{2^j}, a + \frac{h}{2^{j+1}}\Big) \times \Big(b + h - \frac{(1+\xi)h}{2^j}, b + h - \frac{h}{2^{j+1}}\Big).$$

Next, $Q_j = (P_j)^-_{2\xi+1}$, where

$$P_j = \Big(a + \frac{h}{2^{j+1}}, a + \frac{h}{2^j}\Big) \times \Big(b + h - \frac{h}{2^{j+1}}, b + h\Big).$$

Clearly, $\cup_{j=0}^{\infty} P_j \subset Q$ and P_j are pairwise disjoint. Hence, by (3.4),

$$w(T_1) \le \sum_{j=0}^{\infty} w(Q_j) \le c \sum_{j=0}^{\infty} w(P_j) \le cw(Q).$$

The proof is complete. $\square$

The following lemma is a key part of our proof.

Lemma 3.3. *Let $w \in A_p^+$. Then*

$$w\{x : N_r^+ f(x) > \lambda\} \le cw\{x : N_{1/3}^+ f(x) > \lambda/3\} \quad (0 < r < 1/4, \lambda > 0),$$

where the constant c depends only on r and w.

Proof. Set $E_\lambda = \{x : N_r^+ f(x) > \lambda\}$, and let $x \in E_\lambda$. Then there exists $h > 0$ such that $f_{Q^r_{x,h}} > \lambda$. Let $i = i(r)$ be the smallest natural number for which $2^i \ge 4/r$. We divide $Q^r_{x,h}$ into 4^i equal squares. Then there exists at least one of them (denote it by R_x) such that $f_{R_x} > \lambda$.

Consider now the square $P_x = (R_x^-)^-$ (see Figure 3). For any $y \in P_x$ there exists a square $\bar{Q}$ such that y is the left lower corner of $\bar{Q}$, $R_x \subset \bar{Q}^{1/3}_{y,\ell_{\bar{Q}}}$ and $|\bar{Q}| \le 9|R_x|$. Then $f_{\bar{Q}^{1/3}_{y,\ell_{\bar{Q}}}} \ge (4/9) f_{R_x} > 4\lambda/9$. Therefore, for any $y \in P_x$ we have $N^+_{1/3} f(y) > 4\lambda/9$.

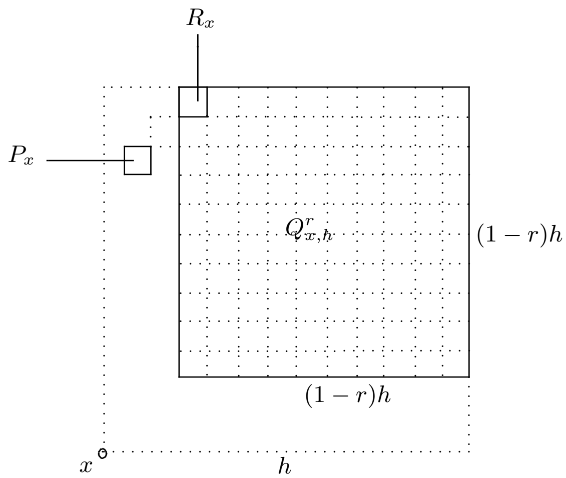

FIGURE 3

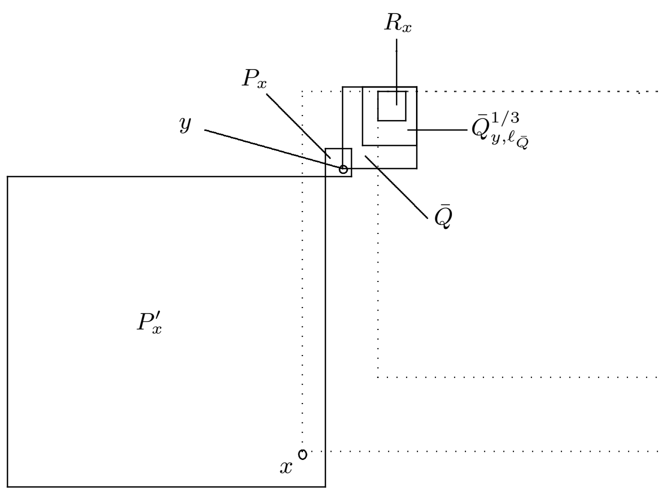

FIGURE 4

It is easy to see that there exists a square P'_x (see Figure 4) and such that

(i) the right upper corner of P'_x coincides with the left lower corner of P_x;
(ii) $x \in \alpha P'_x$, where $\alpha = \alpha(r) < 1$;
(iii) $\ell_{P'_x} \leq \beta \ell_{P_x}$, where $\beta = \beta(r) > 1$.

Let F_x be the convex hull of $P'_x \cup P_x$. Applying to the family $\{F_x\}_{x \in E_\lambda}$ the Besicovitch covering theorem [8, Ch. 1], we get a sequence $\{x_k\}$ such that

(1) $E_\lambda \subset \cup_k F_{x_k}$;

(2) $\sum_k \chi_{F_{x_k}}(x) \le c$.

Therefore, by Lemma 3.2,

$$w(E_\lambda) \le \sum_k w(F_{x_k}) \le c \sum_k w(P_{x_k}) \le cw\{x : N^+_{1/3} f(x) > 4\lambda/9\},$$

which completes the proof. □

Theorem 3.4. *Let $n = 2$. Then $M^+ : L^p_w \to L^{p,\infty}_w$ if and only if $w \in A^+_p$.*

This theorem was proved in [7].

Proof of Theorem 1.5. One can assume that $0 < r < 1/4$. It follows from Lemma 3.1 that

$$N^+_{1/3}(w^{1+\delta})(x) \le c\delta N^+_r\big((M^+w)^{1+\delta}\big)(x) + (N^+_{1/3}w)^{1+\delta}(x),$$

and therefore,

$$N^+_{1/3}(w^{1+\delta})(x) \le c\|w\|^{1+\delta}_{A^-_1}\big(\delta N^+_r\big(w^{1+\delta}\big)(x) + w^{1+\delta}(x)\big) \tag{3.5}$$

(here $A^-_1 = A_1(\mathcal{Q}^+)$).

Let $R_\varepsilon f(x) = \sum_{k=0}^\infty \varepsilon^k (M^+)^k f(x)$. Then $\|R_\varepsilon f\|_{A^-_1} \le \frac{1}{\varepsilon}$. Setting $w = R_\varepsilon(f^{\frac{1}{1+\delta}})$ in (3.5), and denoting $T_{\varepsilon,\delta} f = R_\varepsilon(f^{\frac{1}{1+\delta}})^{1+\delta}$, we get

$$N^+_{1/3}(T_{\varepsilon,\delta} f)(x) \le \frac{c}{\varepsilon^{1+\delta}}\big(\delta N^+_r(T_{\varepsilon,\delta} f)(x) + T_{\varepsilon,\delta} f(x)\big).$$

From this and from Lemma 3.3,

$$\begin{aligned} w\{x : N^+_r(T_{\varepsilon,\delta} f)(x) > \lambda\} &\le c_1 w\Big\{x : N^+_r(T_{\varepsilon,\delta} f)(x) > \frac{\varepsilon^{1+\delta}\lambda}{6c_2\delta}\Big\} \\ &+ c_1 w\Big\{x : T_{\varepsilon,\delta} f(x) > \frac{\varepsilon^{1+\delta}\lambda}{6c_2}\Big\}. \end{aligned} \tag{3.6}$$

Assume now that $f \in L^\infty \cap L^p_w$. Then $N^+_r(T_{\varepsilon,\delta} f) \in L^\infty$, and hence for any $a > 0$,

$$I(a) = \int_a^\infty \lambda^{p-1} w\{x : N^+_r(T_{\varepsilon,\delta} f)(x) > \lambda\} d\lambda < \infty.$$

It follows from (3.6) that

$$I(a) \le c_1 \Big(\frac{6c_2\delta}{\varepsilon^{1+\delta}}\Big)^p I(a\varepsilon^{1+\delta}/6c_2\delta) + c(\varepsilon, \delta)\|T_{\varepsilon,\delta} f\|^p_{L^p_w}.$$

Set now $\delta = \gamma\varepsilon$, where γ is so that $c_1\Big(\frac{6c_2\gamma}{\varepsilon^{\gamma\varepsilon}}\Big)^p \le 1/2$. Then

$$I(a) \le 2c(\varepsilon, \gamma\varepsilon)\|T_{\varepsilon,\gamma\varepsilon} f\|^p_{L^p_w}.$$

Next we note that

$$\|T_{\varepsilon,\gamma\varepsilon}f\|_{L^p_w} \le \Big(\sum_{k=0}^{\infty}(\varepsilon\|M^+\|_{L_w^{p(1+\gamma\varepsilon)}})^k\Big)^{1+\gamma\varepsilon}\|f\|_{L^p_w}.$$

It follows from Theorem 3.4 and from the Marcinkiewicz interpolation theorem that

$$\|M^+\|_{L_w^{p(1+\gamma\varepsilon)}} \le \frac{c}{(\gamma\varepsilon)^{1/p}}.$$

Taking ε so that $c\varepsilon^{1-1/p}/\gamma^{1/p} < 1$, and combining the previous estimates, we obtain

$$I(a) \le c\|f\|^p_{L^p_w}.$$

Letting $a \to 0$, and using that $|f| \le T_{\varepsilon,\delta}f$, we get

$$\|N_r^+ f\|_{L^p_w} \le c\|f\|_{L^p_w}.$$

Finally we note that the restriction $f \in L^\infty$ is easily removed by the Fatou convergence theorem. □

4. Some applications of Theorem 1.2

4.1. Maximal characterization of the A_p condition.

Let

$$M_w f(x) = \sup_{Q\ni x}\frac{1}{w(Q)}\int_Q |f(y)|w(y)dy.$$

In the Introduction we have observed that Muckenhoupt's theorem follows easily from Corollary 1.3. The argument given shows that a weight w satisfies the A_p condition iff w is doubling (i.e., there exists $c > 0$ such that $w(2Q) \le cw(Q)$ for any Q) and

$$Mf(x)^p \le cM_w(|f|^p)(x). \tag{4.1}$$

Here we notice that the A_p condition can be fully characterized in terms of (4.1) only.

Proposition 4.1. *Let w be a weight. Then w satisfies the A_p condition iff inequality (4.1) holds for any $f \in L^1_{loc}(\mathbb{R}^n)$ and for all $x \in \mathbb{R}^n$.*

Remark 4.2. The fact that (4.1) follows from the A_p condition is well-known [4]. However, we have never seen in the literature the converse statement.

Proof of Proposition 4.1. In the one-dimensional case the proof is immediate since the weighted maximal operator M_w is always of weak type $(1,1)$ with respect to w [22], and therefore (4.1) implies the weak type (p,p) of M. It remains to apply Corollary 1.3. In the case $n \ge 2$ we only need to show that (4.1) implies the doubling property of w. Then the same arguments work.

We shall use the notation from Section 3 with an obvious generalization to any dimension. First, we remark that for any cube Q,

$$c_1 w(Q_\xi^-) \le w(Q) \le c_2 w(Q_\xi^-) \quad (\xi > 0). \tag{4.2}$$

Indeed, let x_Q be the "upper right" corner of Q. Then it is easy to see that with $f = \chi_{Q_\xi^-}$ we have $M_w(f)(x_Q) \le w(Q_\xi^-)/w(Q)$, and $M(f)(x_Q) \ge c$. From this and from (4.1) we get the right-hand side of (4.2); the left-hand side can be obtained in a similar way.

Next, observing that $Q_{1/2}^- \subset Q^-$, and combining inequalities in (4.2), we get

$$w(2Q) \le cw((2Q)^-) \le cw(Q_{1/2}^-) \le cw(Q^-) \le cw(Q),$$

which completes the proof. □

4.2. **On the property $A_p(\mathcal{B}) \Rightarrow A_{p-\varepsilon}(\mathcal{B})$.** Let $\mathcal{B}$ be a Buseman-Feller basis (BF-basis). This means that if $B \in \mathcal{B}$ and $x \in B$, then $B \in \mathcal{B}(x)$. Replacing in the definitions of A_p and M_w cubes by sets $B \in \mathcal{B}$ we get the $A_p(\mathcal{B})$ condition and the maximal operator $M_{\mathcal{B},w}$. It is easy to see that the $A_p(\mathcal{B})$ condition is necessary for $M_{\mathcal{B}}$ to be bounded on L_w^p. Next, it was shown by B. Jawerth [9] that if

$$A_p(\mathcal{B}) \Rightarrow M_{\mathcal{B},w} : L_w^r \to L_w^r \quad (r > 1), \tag{4.3}$$

then $M_{\mathcal{B}}$ is bounded on L_w^p. Therefore, by (ii) $\Rightarrow$ (iii) of Theorem 1.2 we have that if $\mathcal{B}$ satisfies Stein's property and (4.3) holds, then $A_p(\mathcal{B}) \Rightarrow A_{p-\varepsilon}(\mathcal{B})$.

Consider, for example, the Cordoba basis $\mathcal{R}_\Phi$, where $\mathcal{R}_\Phi(x)$ consists of all rectangles in $\mathbb{R}^n$ containing x with dimensions $s_1 \times \cdots \times s_{n-1} \times \Phi(s_1, \dots, s_{n-1})$. Here Φ is a nonnegative continuous function, monotone in each variable and satisfying

$$\Phi(s_1, \dots, s_{j-1}, 0, s_{j+1}, \dots, s_{n-1}) = 0 \quad (1 \le j \le n-1),$$

and $\Phi(s_1, \dots, s_{n-1}) \approx \Phi(2s_1, \dots, 2s_{n-1})$. Clearly, $\mathcal{R}_\Phi$ is a BF-basis. Next, using properties of Φ, it can be easily shown that $\mathcal{R}_\Phi$ satisfies Stein's property (it is enough to consider a "dyadic grid" with respect to a given rectangle R and then use the same argument as in [23]). Finally, (4.3) for $\mathcal{B} = \mathcal{R}_\Phi$ was proved in [10]. Therefore, we have that $A_p(\mathcal{R}_\Phi) \Rightarrow A_{p-\varepsilon}(\mathcal{R}_\Phi)$. In the case $n = 3$ and $\Phi(s,t) = st$ this result is contained in [6].

4.3. **Lorentz-Shimogaki Theorem.** Given a measurable function f, the local maximal function $m_\lambda f$ is defined by

$$m_\lambda f(x) = \sup_{Q \ni x} (f\chi_Q)^*(\lambda|Q|) \quad (0 < \lambda < 1),$$

where f^* denotes the non-increasing rearrangement of f.

In a recent paper [13], the authors proved that the maximal operator M is bounded on a quasi-Banach function space X iff

$$\alpha_X \equiv \lim_{\lambda\to 0}\frac{\log\|m_\lambda\|_X}{\log\frac{1}{\lambda}} < 1.$$

This result is a generalization of the classical Lorentz-Shimogaki theorem [2, p. 154], since it is shown in [13] that in the case when X is rearrangement-invariant the index α_X coincides with the upper Boyd index $\bar{\alpha}_X$.

As in the classical case, the part showing that the boundedness of M implies $\alpha_X < 1$ is more complicated. Among other ingredients, the proof in [13] was based on the theory of submultiplicative functions. Here we remark that this part follows immediately from Theorem 1.2. Indeed, by Chebyshev's inequality,

$$(f\chi_Q)^*(\lambda|Q|) = (|f|^r\chi_Q)^*(\lambda|Q|)^{1/r} \le (1/\lambda)^{1/r}\Big(\frac{1}{|Q|}\int_Q |f|^r\Big)^{1/r}.$$

From this and from (ii) $\Rightarrow$ (iii) of Theorem 1.2 we get $\|m_\lambda\|_X \le c(1/\lambda)^{1/r}$, and therefore $\alpha_X \le 1/r$ for some $r > 1$.

4.4. **Ariño-Muckenhoupt Theorem.** Given a non-negative function w on $(0,\infty)$, the Lorentz space $\Lambda_p(w)$ consists of all measurable f on $\mathbb{R}^n$ for which

$$\|f\|_{\Lambda_p(w)} \equiv \Big(\int_0^\infty f^*(t)^p w(t)dt\Big)^{1/p} < \infty.$$

In [1], M.A. Ariño and B. Muckenhoupt proved that M is bounded on $\Lambda_p(w), 1\le p<\infty$, iff w satisfies the following B_p condition:

$$\int_t^\infty \frac{w(\tau)}{\tau^p}d\tau \le \frac{c}{t^p}\int_0^t w(\tau)d\tau \quad (t>0).$$

Note that $(Mf)^*(t) \asymp f^{**}(t) = \frac{1}{t}\int_0^t f^*(\tau)d\tau$ [2, p. 122], and hence the boundedness of M on $\Lambda_p(w)$ means that

$$\|f^{**}\|_{L^p_w} \le c\|f^*\|_{L^p_w}. \tag{4.4}$$

The key ingredient of the proof in [1] was the property $B_p \Rightarrow B_{p-\varepsilon}$. Later, C.J. Neugebauer [18] found a direct and simpler proof of (4.4); the property $B_p \Rightarrow B_{p-\varepsilon}$ was then deduced as a corollary.

Here we notice that exactly as in the case of A_p weights, (ii) $\Rightarrow$ (iii) of Theorem 1.2 yields $B_p \Rightarrow B_{p-\varepsilon}$. In order to apply (ii) $\Rightarrow$ (iii) we only should mention the well-known fact saying that if M is bounded on $\Lambda_p(w)$, then $\Lambda_p(w)$ is a Banach space (because the operator $f\to f^{**}$ is subadditive [2, p. 53]).

For the sake of completeness we outline here a different elementary proof of the boundedness of M on $\Lambda_p(w)$. Let $H\varphi(t)=\frac{1}{t}\int_0^t\varphi(\tau)d\tau$. Then the B_p condition yields

$$\begin{aligned}\int_0^\infty (H\varphi)^p(t)w(t)dt &= \int_0^\infty (tH\varphi)^p(t)'\int_t^\infty \frac{w(\tau)}{\tau^p}d\tau dt\\ &\le c\int_0^\infty (tH\varphi)^p(t)'\frac{1}{t^p}\int_0^t w(\tau)d\tau dt\\ &= cp\int_0^\infty\Big(\int_t^\infty (H\varphi)^{p-1}(\tau)\frac{\varphi(\tau)}{\tau}d\tau\Big)w(t)dt. \qquad (4.5)\end{aligned}$$

Let $\varphi(t)=f^*(t)-f^*(2t)$. Then

$$\begin{aligned}\int_t^\infty (H\varphi)^{p-1}(\tau)\frac{\varphi(\tau)}{\tau}d\tau &\le f^{**}(t)^{p-1}\int_t^\infty\frac{f^*(\tau)-f^*(2\tau)}{\tau}d\tau\\ &\le f^{**}(t)^{p-1}f^*(t),\end{aligned}$$

and applying (4.5) gives

$$\int_0^\infty \big(f^{**}(t)-f^{**}(2t)\big)^p w(t)dt\le c\int_0^\infty f^{**}(t)^{p-1}f^*(t)w(t)dt.$$

Hence, using that $f^{**}(t)-f^*(t)\le 2(f^{**}(2t)-f^{**}(t))$, we get

$$\|f^{**}\|_{L^p_w}\le\|f^{**}-f^*\|_{L^p_w}+\|f^*\|_{L^p_w}\le c\Big(\int_0^\infty f^{**}(t)^{p-1}f^*(t)w(t)dt\Big)^{1/p}$$

From this and Hölder's inequality we obtain (4.4).

We refer to a recent work [3] for numerous extensions and variants of the Ariño-Muckenhoupt theorem.

References

[1] M.A. Ariño and B. Muckenhoupt, *Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for nonincreasing functions*, Trans. Amer. Math. Soc. **320** (1990), no. 2, 727–735.

[2] C. Bennett and R. Sharpley, *Interpolation of operators*, Academic Press, New York, 1988.

[3] M.J. Carro, J.A. Raposo and J. Soria, *Recent developments in the theory of Lorentz spaces and weighted inequalities*, Memoirs Amer. Math. Soc. **187** (2007).

[4] R.R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. **51** (1974), 241–250.

[5] J. Duoandikoetxea, *Fourier Analysis*, Grad. Studies Math. **29**, Amer. Math. Soc., Providence, 2000.

[6] R. Fefferman, *Some weighted norm inequalities for Cordoba's maximal function*, Amer. J. Math. **106** (1984), 1261–1264.

[7] L. Forzani, F.J. Martín-Reyes and S. Ombrosi, *Weighted inequalities for the two-dimensional one-sided Hardy-Littlewood maximal function*, preprint. Available at `http://webpersonal.uma.es/~MARTIN_REYES/preprints.htm`

[8] M. de Guzman, *Differentiation of integrals in* $\mathbb{R}^n$, Lecture Notes in Math., 481, Springer-Verlag 1975.

[9] B. Jawerth, *Weighted inequalities for maximal operators: Linearization, localization and factorization*, Amer. J. Math. **108** (1986), 361–414.

[10] B. Jawerth and A. Torchinsky, *The strong maximal function with respect to measures*, Studia Math. **80** (1984), 261–285.

[11] N.J. Kalton, N.T. Peck and J.W. Roberts, *An F-space sampler*, London Mathematical Society Lecture Note Series, **89**, Cambridge University Press, 1984.

[12] A.K. Lerner, S. Ombrosi and C. Pérez, *Sharp* A_1 *bounds for Calderón-Zygmund operators and the relationship with a problem of Muckenhoupt*, Internat. Math. Res. Not. (2008).

[13] A.K. Lerner and C. Pérez, *A new characterization of the Muckenhoupt* A_p *weights through an extension of the Lorentz-Shimogaki Theorem*, Indiana Univ. Math. J., **56** (2007) no. 6, 2697-2722.

[14] F.J. Martín-Reyes, *New proofs of weighted inequalities for the one-sided Hardy-Littlewood maximal functions*, Proc. Amer. Math. Soc. **117** (1993), no. 3, 691–698.

[15] F.J. Martín-Reyes, *On the one-sided Hardy-Littlewood maximal function in the real line and in dimensions greater than one*, Fourier analysis and partial differential equations (Miraflores de la Sierra, 1992), 237–250, Stud. Adv. Math. CRC, Boca Raton, FL, 1995.

[16] F.J. Martin-Reyes and A. de la Torre, *Two weight norm inequalities for fractional one-sided maximal operators*, Proc. Amer. Math. Soc. **117** (1993), no. 2, 483–489.

[17] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226.

[18] C.J. Neugebauer, *Weighted norm inequalities for average operators of monotone functions*, Publ. Mat. **35** (1991), 429–447.

[19] S. Ombrosi, Weak weighted inequalities for a dyadic one-sided maximal function in $\mathbb{R}^n$, Proc. Amer. Math. Soc. **133** (2005), no. 6, 1769–1775.

[20] J.L. Rubio de Francia, *Factorization theory and* A_p *weights*, Amer. J. Math. **106** (1984), 533–547.

[21] E. Sawyer, *Weighted inequalities for the one-sided Hardy-Littlewood maximal functions*, Trans. Amer. Math. Soc. **297** (1986), 53–61.

[22] P. Sjögren, *A remark on the maximal function for measures in* $\mathbb{R}^n$, Amer. J. Math. **105** (1983), 1231–1233.

[23] E.M. Stein, *Note on the class* $L \log L$, Studia Math. **32** (1969), 305–310.

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