WEAK WEIGHTED INEQUALITIES FOR A DYADIC ONE-SIDED MAXIMAL FUNCTION IN $\mathbb{R}^n$

SHELDY OMBROSI

Abstract. In this note we introduce a dyadic one-sided maximal function defined as $M^{+,d}f(x) = \sup_{Q\ dyadic:x\in Q} \frac{1}{|Q|}\int_{Q^+}|f|$, where Q^+ is a certain cube associated with the dyadic cube Q and $f \in L^1_{loc}(\mathbb{R}^n)$. We characterize the pair of weights (w,v) for which the maximal operator $M^{+,d}$ applies $L^p(v)$ into weak-$L^p(w)$ for $1 \le p < \infty$.

1. Introduction, notation and main results

In [10] E. Sawyer introduced the class of weights A_p^+, which characterizes the pair of weights (w,v) such that the one-sided Hardy-Littlewood maximal operator apply $L^p(v)$ into weak-$L^p(w)$. These classes of weights and their associated theory have been the subject of much study. Different proofs of Sawyer´s results and more general extensions are given in the papers of F. J. Martín-Reyes, Ortega-Salvador and A. de la Torre [6], [4] and [5]. In [1] H. Aimar, L. Forzani and F. J. Martín-Reyes proved the existence of one-sided singular integral operators, and L. de Rosa and C. Segovia have developed great part of the theory of one-sided weighted Hardy spaces (see [8] and [9]). However, as far as the author knows, the theory of one-sided weights has been developed **only** in $\mathbb{R}$.

The results in this note try to give a first step to overcome that limitation. We will define a dyadic one-sided maximal function in $\mathbb{R}^n$ and we characterize the pair of weights (w,v) such that the operator associated with this maximal function applies $L^p(v)$ into weak-$L^p(w)$ (Theorem 1.2 below).

If $I=[a,b)$ is a bounded interval we denote $I^+=[b,2b-a)$ and $I^-=[2a-b,a)$. If $Q=I_1\times I_2\times ...\times I_n$ is a cube in $\mathbb{R}^n$, we denote $Q^+=I_1^+\times I_2^+\times ...\times I_n^+$, and $Q^-=I_1^-\times I_2^-\times ...\times I_n^-$.

Given $f\in L^1_{loc}(\mathbb{R}^n)$, we define the dyadic one-sided maximal function $M^{+,d}f(x)$ as

$$M^{+,d}f(x)=\sup_{Q\ dyadic:x\in Q}\frac{1}{|Q|}\int_{Q^+}|f|,$$

In a similar way $M^{-,d}f(x)=\sup_{Q\ dyadic:x\in Q}\frac{1}{|Q|}\int_{Q^-}|f|$. In dimension one, this maximal function and, more generally, dyadic one-sided fractional maximal functions were defined in [5].

Let $x\in\mathbb{R}^n$, $x=(x_1,x_2,...,x_n)$, and let h be a positive real number. We denote $Q_{x,h}=[x_1,x_1+h)\times[x_2,x_2+h)\times...\times[x_n,x_n+h)$, $Q_{x,h^-}=[x_1-h,x_1)\times[x_2-h,x_2)\times...\times[x_n-h,x_n)$, and $Q^{\frac{1}{2}}_{x,h}=[x_1+\frac{h}{2},x_1+h)\times[x_2+\frac{h}{2},x_2+h)\times...\times[x_n+\frac{h}{2},x_n+h)$.

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With this notation, we define the following maximal functions

$$M^{+}f(x)=\sup_{h>0}\frac{1}{|Q_{x,h}|}\int_{Q_{x,h}}|f|\,,\ \ M^{-}f(x)=\sup_{h>0}\frac{1}{\left|Q_{x,h^{-}}\right|}\int_{Q_{x,h^{-}}}|f|\,,\text{ and}$$

$$N^{+}f(x)=\sup_{h>0}\frac{1}{\left|Q_{x,h}^{\frac{1}{2}}\right|}\int_{Q_{x,h}^{\frac{1}{2}}}|f|\,,$$

In dimension $n=1$, the maximal functions $M^{+,d}f\left(x\right)$, $N^{+}f(x)$, and $M^{+}f\left(x\right)$ are pointwise equivalents (see [5]), this fact is not true if $n>1$.

Let $Q=[a_1,b_1)\times[a_2,b_2)\times...\times[a_n,b_n)$, with $b_i-a_i=h$, and let s be a positive real number. We denote $(Q)^{s,+}=[a_1,a_1+sh)\times[a_2,a_2+sh)\times...\times[a_n,a_n+sh)$, and $(Q)^{s,-}=[b_1-sh,b_1)\times[b_2-sh,b_2)\times...\times[b_n-sh,b_n)$. We observe that $\left|(Q)^{s,+}\right|=\left|(Q)^{s,-}\right|=s^{n}\left|Q\right|$ and $Q=(Q)^{1,+}=(Q)^{1,-}$.

Remark 1.1. *Let R and Q be cubes such that $R^{+}\subset Q^{+}$, then $R\subset(Q)^{2,+}$.*

As usual, a weight $w(x)$ is a measurable and non-negative function. If $E\subset\mathbb{R}^{n}$ is a Lebesgue measurable set, we denote its w-measure by $w(E)=\int_{E}w(t)dt$.

We will say that a pair of weights (w,v) satisfies the condition $A_{p}^{+}\left(\mathbb{R}^{n}\right)$, (or (w,v) belongs to the class $A_{p}^{+}\left(\mathbb{R}^{n}\right)$) $p>1$, if there exists a constant C such that for every cube Q,

$$\int_{Q}w\left(\int_{Q^{+}}v^{-\frac{1}{p-1}}\right)^{p-1}\leq C\left|Q\right|^{p}. \tag{1}$$

The condition $A_{1}^{+}\left(\mathbb{R}^{n}\right)$ is given by

$$M^{-}w(x)\leq Cv(x)\ \text{ a.e.} \tag{2}$$

Interesting gemetric properties of this classes of weights were obtained by Martín-Reyes in [3]

We define similar classes of weights on dyadic cubes. We will say that a pair of weights (w,v) satisfies the condition $A_{p}^{+,d}\left(\mathbb{R}^{n}\right)$ $(p>1)$ if (1) holds for every dyadic cube Q, and the condition $A_{1}^{+,d}$ is given by $M^{-,d}w(x)\leq Cv(x)$ a. e.

It is clear that if $w\geq0$ is a nondecreasing function in each coordinate, the pair (w,w) satisfies the condition $A_{p}^{+}\left(\mathbb{R}^{n}\right)$ and therefore $A_{p}^{+,d}\left(\mathbb{R}^{n}\right)$. Thus, these conditions are weaker than the Muckenhoupt´s condition $A_{p}\left(\mathbb{R}^{n}\right)$ introduced in [7], which characterizes the pair of weights such that the Hardy-Littlewood maximal operator applies $L^{p}\left(v\right)$ into weak-$L^{p}\left(w\right)$.

The next theorem solves a conjecture of A. de la Torre and F. J. Martín-Reyes.

Theorem 1.2. *Let w and v be weights on $\mathbb{R}^{n}$ and let $1\leq p<\infty$. Then, the following conditions are equivalent:*

- *(a) The pair of weights (w,v) satisfies the condition $A_{p}^{+,d}\left(\mathbb{R}^{n}\right)$*
- *(b) The maximal operator $M^{+,d}$ is of weak type (p,p) with respect to the pair (w,v), that is: there is a constant C such that for every function $f\in L_{loc}^{1}\left(\mathbb{R}^{n}\right)$ and every $\lambda>0$*

$$w\left(\left\{x:M^{+,d}f(x)>\lambda\right\}\right)\leq C\lambda^{-p}\int_{\mathbb{R}^{n}}|f|^{p}\,v$$

Theorem 1.3. *Let w and v be two weights on $\mathbb{R}^n$ and let $1 \leq p < \infty$. If (w,v) belongs to the class $A_p^+(\mathbb{R}^n)$ then the maximal operator N^+f is of weak type (p,p) with respect to the pair (w,v).*

2. Proof of the results

Let f be an integrable and non-negative function and let λ be a positive real number. We denote $\Omega_\lambda = \{x : M^{+,d}f(x) > \lambda\}$. It is not hard to see that there exists a disjoint collection of dyadic cubes $\{Q_j\}$ such that

$$\cup_j Q_j = \Omega_\lambda \text{ and for each } j,\ f_{Q_j^+} = \frac{1}{|Q_j|}\int_{Q_j^+} f > \lambda.$$

In general, the associated collection $\{Q_j^+\}$ is not disjoint. This fact is the main difficulty that appearing in our proof of Theorem 1.2. However with the next lemma we overcome that difficulty.

Lemma 2.1. *Let $1 \leq p < \infty$. Let (w,v) be a pair of weights satisfying the condition $A_p^{+,d}(\mathbb{R}^n)$. Let μ be a positive real number. If $\{Q_j\}_{j\in\Gamma_\mu}$ is a disjoint collection of dyadic cubes (Γ_μ denotes the family of indexes j) such that for each $j \in \Gamma_\mu$, $\mu < f_{Q_j^+} \leq 2\mu$, then*

$$\sum_{j\in\Gamma_\mu} w(Q_j) \leq \frac{2^{p+n+2}C}{\mu^p}\int_{\cup_{j\in\Gamma_\mu} Q_j^+} |f|^p\, v,$$

where C is the smallest constant that we can take in the condition $A_p^{+,d}(\mathbb{R}^n)$ for the pair (w,v).

Proof. For each integer $m \geq 0$, we define the subfamily of indexes i_m in Γ_μ as

$$i_m = \left\{j \in \Gamma_\mu :\ \text{there exist exactly } m \text{ cubes } Q_s^+ \text{: } Q_j^+ \subsetneqq Q_s^+ \text{ with } s \in \Gamma_\mu\right\},$$

and we denote

$$\sigma_m = \cup_{j\in i_m} Q_j^+.$$

Remarks:

(1) It is clear that $\Gamma_\mu = \cup_{m\geq 0} i_m$. Furthermore, if j_1, j_2 belong to i_m then $Q_{j_1}^+ \cap Q_{j_2}^+ = \emptyset$, and therefore

$$\sum_{j\in i_m}\int_{Q_j^+} f = \int_{\sigma_m} f.$$

(2) If $j \in i_{m+1}$ $(m \geq 0)$ then for each integer k, $0 \leq k \leq m$, there is only one $j_k \in i_k$ such that $Q_j^+ \subset Q_{j_k}^+$. Thus $\sigma_{m+1} \subset \sigma_m$ and $\int_{\sigma_{m+1}} f \leq \int_{\sigma_m} f$.

(3) We fix a non negative integer m_0 and $j_0 \in i_{m_0}$. If $Q_j^+ \subsetneqq Q_{j_0}^+$ then $j \in i_m$ where $m > m_0$, and moreover, by remark 1.1 we have that $Q_j \subset (Q_{j_0})^{2,+}$.

Taking into account the last observation and since the cubes Q_j are disjoint we have

$$\cup_{m>m_0} \cup_{\substack{j\in i_m:\\ Q_j^+\subsetneqq Q_{j_0}^+}} Q_j \subset (Q_{j_0})^{2,+},$$

and

$$\sum_{m>m_0}\sum_{\substack{j\in i_m:\\ Q_j^+\subsetneq Q_{j_0}^+}}|Q_j|\leq 2^n\,|Q_{j_0}|\,.$$

Therefore, since for every $j\in\Gamma_\mu$, $\mu<f_{Q_j^+}\leq 2\mu$, we get

$$\sum_{m>m_0}\int_{\sigma_m\cap Q_{j_0}^+}f=\sum_{m>m_0}^{\infty}\sum_{\substack{j\in i_m:\\ Q_j^+\subsetneq Q_{j_0}^+}}\int_{Q_j^+}f$$

$$\leq 2\mu\sum_{m>m_0}\sum_{\substack{j\in i_m:\\ Q_j^+\subsetneq Q_{j_0}^+}}|Q_j|\leq 22^n\mu\,|Q_{j_0}|<2^{n+1}\int_{Q_{j_0}^+}f.$$

Thus, we have that $\sum_{m=m_0+1}^{m_0+2^{n+2}}\int_{\sigma_m\cap Q_{j_0}^+}f<2^{n+1}\int_{Q_{j_0}^+}f$. Then, there is a index m, $m_0+1\leq m\leq m_0+2^{n+2}$, such that $\int_{\sigma_m\cap Q_{j_0}^+}f<\frac{1}{2}\int_{Q_{j_0}^+}f$, and since $\int_{\sigma_{m+1}\cap Q_{j_0}^+}f\leq\int_{\sigma_m\cap Q_{j_0}^+}f$, we have that, if $j_0\in m_0$,

$$\int_{Q_{j_0}^+\cap\sigma_{m_0+2^{n+2}}}f<\frac{1}{2}\int_{Q_{j_0}^+}f,$$

which implies that

$$\int_{Q_{j_0}^+-\sigma_{m_0+2^{n+2}}}f>\frac{1}{2}\int_{Q_{j_0}^+}f.$$

From the last inequality and since $\mu<f_{Q_{j_0}^+}$, we get

$$\frac{1}{|Q_{j_0}|}\int_{Q_{j_0}^+-\sigma_{m_0+2^{n+2}}}f>\frac{\mu}{2}. \tag{3}$$

Now we will estimate $\sum_{j\in\Gamma_\mu}w(Q_j)$. We suppose $p>1$. From (3), Hölder´s inequality, and by condition $A_p^{+,d}(\mathbb{R}^n)$, we have

$$\begin{aligned}\sum_{j\in\Gamma_\mu}w(Q_j) &= \sum_{m=0}^{\infty}\sum_{j\in i_m}w(Q_j)\leq\left(\frac{2}{\mu}\right)^p\sum_{m=0}^{\infty}\sum_{j\in i_m}w(Q_j)\left(\frac{1}{|Q_j|}\int_{Q_j^+-\sigma_{m+2^{n+2}}}f\right)^p\\ &\leq \left(\frac{2}{\mu}\right)^p\sum_{m=0}^{\infty}\sum_{j\in i_m}w(Q_j)\left[v^{-\frac{1}{p-1}}\left(Q_j^+\right)\right]^{p-1}\frac{1}{|Q_j|^p}\int_{Q_j^+-\sigma_{m+2^{n+2}}}f^pv\\ &\leq \frac{2^pC}{\mu^p}\sum_{m=0}^{\infty}\int_{\sigma_m-\sigma_{m+2^{n+2}}}f^pv. \qquad (4)\end{aligned}$$

(If $p=1$ we simply observe that $\frac{w(Q_j)}{|Q_j|}\int_{Q_j^+-\sigma_{m+2^{n+2}}}f\leq\int_{Q_j^+-\sigma_{m+2^{n+2}}}f\;M^{-,d}w$, and by condition $A_1^{+,d}(\mathbb{R}^n)$ we can obtain the previous estimate also in this case).

Then, by (4) and since

$$\begin{aligned}\sum_{m=0}^{\infty}\int_{\sigma_m-\sigma_{m+2^{n+2}}} f^p v &= \sum_{k=0}^{2^{n+2}-1}\sum_{m=0}^{\infty}\int_{\sigma_{2^{n+2}m+k}-\sigma_{2^{n+2}(m+1)+k}} f^p v \\ &= \sum_{k=0}^{2^{n+2}-1}\int_{\sigma_k} f^p v \le 2^{n+2}\int_{\sigma_0} f^p v,\end{aligned}$$

we obtain the lemma. ■

Proof of Theorem 1.2 . We will prove that $(a) \Rightarrow (b)$ since $(b) \Rightarrow (a)$ can been obtained, with simple changes, by a standard argument (e.g., see pp. 390 in [2]).

Assume that $(w, v) \in A_p^{+,d}(\mathbb{R}^n)$.We will estimate

$$w\left(\{x : M^{+,d} f(x) > \lambda\}\right) = w\left(\Omega_\lambda\right).$$

Recall that we can write $\Omega_\lambda = \cup_j Q_j$, where $f_{Q_j^+} > \lambda$.

For each non-negative integer k, we consider the subfamily of indexes $c_k = \left\{j : 2^k\lambda < f_{Q_j^+} \le 2^{k+1}\lambda\right\}$. Then, we can apply Lemma 2.1 to the collection $\{Q_j\}_{j\in c_k}$, with $\mu = 2^k\lambda$, getting

$$\sum_{j\in c_k} w(Q_j) \le \frac{2^{p+n+2}C}{(2^k\lambda)^p}\int_{\mathbb{R}^n} f^p v.$$

Therefore

$$\begin{aligned}w\left(\Omega_\lambda\right) &= \sum_j w(Q_j) = \sum_{k=0}^{\infty}\sum_{j\in c_k} w(Q_j) \\ &\le \sum_{k=0}^{\infty}\frac{2^{p+n+2}C}{(2^k\lambda)^p}\int_{\mathbb{R}^n} f^p v = \frac{2^{p+n}8C}{\lambda^p}\int_{\mathbb{R}^n} f^p v.\end{aligned}$$

which proves that $(a) \Rightarrow (b)$. ■

Proof of Theorem 1.3 . Fix an integrable and non-negative function f. We know that

$$\{x : M^{+,d} f(x) > 4^{-n}\lambda\} = \cup_j Q_j, \tag{5}$$

where $\{Q_j\}_j$ is a disjoint collection of dyadic cubes $\{Q_j\}_j$ such that for each j, $f_{Q_j^+} > 4^{-n}\lambda$. We will prove that

$$\{x : N^+ f(x) > \lambda\} \subset \cup_j \left(Q_j\right)^{4,-}. \tag{6}$$

If $x \in \{x : N^+ f(x) > \lambda\}$ there exists $h > 0$ such that $f_{Q_{x,h}^{\frac{1}{2}}} > \lambda$. Let k be the integer such that $2^{k+1} \le h < 2^{k+2}$. Then, there exist at most 4^n dyadic cubes R such that $R \cap Q_{x,h}^{\frac{1}{2}} \neq \emptyset$ and $|R| = 2^{kn}$, and therefore one of them satisfies that $\int_R f \ge 4^{-n}\int_{Q_{x,h}^r} f$. Then, since $|R| = 2^{kn} \le \left|Q_{x,h}^{\frac{1}{2}}\right|$, we have that

$$\frac{1}{|R|}\int_R f \ge \frac{4^{-n}}{|R|}\int_{Q_{x,h}^r} f \ge \frac{4^{-n}}{\left|Q_{x,h}^{\frac{1}{2}}\right|}\int_{Q_{x,h}^{\frac{1}{2}}} f > 4^{-n}\lambda,$$

and therefore the dyadic cube $R^- \subset \{x : M^{+,d} f(x) > 4^{-n}\lambda\}$. Thus there exists a maximal dyadic cube Q_j in the collection in (5) such that $R^- \subset Q_j$, thus $(R^-)^{4,-} \subset (Q_j)^{4,-}$ and since $x \in (R^-)^{4,-}$ we obtain (6).

Now, by condition $A_p^+(\mathbb{R}^n)$, we have that

$$\int_{(Q_j)^{4,-}} w \left(\int_{(Q_j^+)^{4,+}} v^{-\frac{1}{p-1}} \right)^{p-1} \leq C\,|Q_j|^p, \text{ if } p > 1, \text{ and}$$

$$\frac{1}{|Q_j|} \int_{(Q_j)^{4,-}} w \leq C\, M^- w(x) \leq C v(x) \text{ for a.e } x \in Q_j^+, \text{ if } p = 1.$$

Thus, arguing as in the proof of Lemma 2.1 and taking into account the last inequalities, we can prove that

$$\sum_{j: 2^{k-2n}\lambda < f_{Q_j^+} \leq 2^{k-2n+1}\lambda} w\left((Q_j)^{4,-}\right) \leq \frac{C}{(2^k\lambda)^p} \int_{\mathbb{R}^n} |f|^p\, v.$$

Then, following the reasoning as in the proof of Theorem 1.2 we can obtain the theorem. ■

Final Remark

We have already observed that if $n > 1$ the maximal functions $N^+ f(x)$ and $M^+ f(x)$ are not pointwise equivalents. However, we believe that it is possible to prove that if the pair of weights (w, v) satisfies the condition $A_p^+(\mathbb{R}^n)$ then the maximal operator M^+ is of weak type (p, p) with respect to (w, v).

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Sheldy Ombrosi.

mail adress:
Depto. de Matemática, Universidad Nacional del Sur,
Avenida Alem 1253, Bahía Blanca, Buenos Aires, ARGENTINA.
e-mail: sombrosi@uns.edu.ar